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EQUATIONAL LOGIC OF PARTIAL PREDICATES STABLE UNDER TOLERANCE RELATIONS

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Tolerance relations are now widely used in informatics. Partial predicates defined over a set with a tolerance relation that cannot have different values on tolerant elements are called tolerance-stable. In the paper algebra of such predicates has been defined and the corresponding infinitary equational logic has been constructed. The soundness and completeness of this logic have been proved.

KEY WORDS: tolerance relations, equational logic, partial predicate, tolerance stability.

ЭКВАЦИОНАЛЬНАЯ ЛОГИКА ЧАСТИЧНЫХ ПРЕДИКАТОВ, УСТОЙЧИВЫХ ПРИ УСЛОВИЯХ ТОЛЕРАНТНОСТИ

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В настоящее время условия толерантности широко используются в информатике. Частичные предикаты, определенные на множестве с условием толерантности, которые не могут иметь различные значения на толерантных элементах, называются толерантно-устойчивыми. В этой работе определена алгебра таких предикатов и построена соответствующая инфинитарная эквациональная логика. Доказаны правильность и полнота рассматриваемой логики.

КЛЮЧЕВЫЕ СЛОВА: условия толерантности, эквациональная логика, частичные предикаты, толерантная устойчивость.

ЕКВАЦІОНАЛЬНА ЛОГІКА ЧАСТКОВИХ ПРЕДИКАТІВ, СТІЙКИХ ЗА УМОВАМИ ТОЛЕРАНТНОСТІ

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Останнім часом умови толерантності широко використовуються в інформатиці. Часткові предикати, визначені на множині з умовою толерантності, які не можуть мати різні значення на толерантних елементах, називаються толерантно-стійкими. В роботі визначена алгебра таких предикатів і побудована відповідна інфінітарна екваціональна логіка. Доведені правильність і повнота розглянутої логіки.

КЛЮЧОВІ СЛОВА: умови толерантності, екваціональна логіка, часткові предикати, толерантна стійкість.

1. Introduction.

Tolerance relations are now widely used in informatics. Partial predicates defined over a set with a tolerance relation that cannot have different values on tolerant elements are called tolerance-stable. In the paper we defined an algebra of such predicates and constructed corresponding infinitary equational logic. The soundness and completeness of this logic were proved.

Formal logic is widely used in different areas of informatics, cognitive science, artificial intelligence, linguistics etc. Numerous results were obtained (see, for example, multi-volume editions [1, 2]). The diversity of approaches and results is explained, in particular, by the diversity of practical areas of usage of information technologies. When we speak about logics for reasoning in such areas (domains) we should distinguish general (universal) logics and special (singular) logics. General logics are based on domain models capturing their essential properties while special logics are based on some particular domain property. Among particular properties we can mention such binary relations as equivalence and partial order. Domains with such relations were intensively studied for years. A tolerance relation (reflexive and symmetric relation) lacks thorough investigation though this relation is now used more intensively in domains with incomplete and imprecise information (image recognition, machine learning, clusterization, decision making, knowledge bases, etc. (see, for example [3–7]).

In this paper we will study special predicates defined on a set with a tolerance relation. Such predicates (called tolerance-stable), if defined on tolerant elements, should yield on them the same Boolean values. We will define algebras of tolerance-stable predicates and construct a system of identical

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transformations. In other words, an infinitary equational logic of tolerance-stable predicates will be constructed.

2. Formulation of the problem. Let D be an arbitrary set and $\approx \in D \times D$ be a tolerance relation (reflexive and symmetric binary relation on D). The pair $<D, \approx>$ is called *tolerated set*. Let $Bool = \{T, F\}$ be the set of Boolean values, and let $P = D \xrightarrow{p} Bool$ be the set of partial predicates. To indicate partiality or totality of mappings we use arrows $\xrightarrow{p}$ or $\xrightarrow{t}$ respectively. For $p \in P$, $d \in D$, and $b, b' \in Bool$, the expression $p(d)\downarrow$ means that a predicate p is undefined on d, $p(d)\downarrow$ means that p is defined on d, and $p(d) = b$ means that p is defined on d with a value b.

Definition 1. A predicate $p \in P$ is called *tolerance-stable* if, for all $d, d' \in D$ and $b \in Bool$ such that $d \approx d$, $p(d)\downarrow = b$ and $p(d')\downarrow = b'$, we have that $b = b$. We denote the class of tolerance-stable predicates by *PTS*.

Problem under investigation is the following: *given a tolerated set, define an algebra of tolerance-stable predicates and construct an equational logic of such predicates.*

We start with the study of tolerated sets. In the sequel we assume that D is a tolerated set with a tolerance relation $\approx$.

2. Properties of tolerated sets.

Definition 2. Elements $a, b \in D$ are called *intolerant* (denoted $a \not\approx b$) if $(a, b) \notin \approx$.

It is clear that the intolerance relation is symmetric.

Definition 3. Let $A, B \subseteq D$. Sets A and B are called *intolerant* (denoted $A \not\approx B$) if $A \times B \cap \approx = \varnothing$.

In other words, sets A and B are intolerant if there are no $a \in A$ and no $b \in B$ such that $a \approx b$.

Definition 4. Let $d \in D$. The set $d^{\#} = \{a \mid a \not\approx d, a \in D\}$ is called the *intolerant complement* of d.

Definition 5. Let $A \subseteq D$. The set $A^{\#} = \{d \mid \forall a \in A(d \not\approx a), d \in D\}$ is called the *intolerant complement* of A.

From the definitions follows that $A^{\#} = \cap\{a^{\#} \mid a \in A\}$.

Lemma 1. Let $A, B \subseteq D$ and let $A \not\approx B$. Then $A \cap B = \varnothing$, $A^{\#} \supseteq B$, and $A \subseteq B^{\#}$.

The proof is trivial.

3. Properties of tolerance-stable predicates. For investigating tolerance-stable predicates, along with function-theoretic methods, we will also use set-theoretic methods, representing each predicate by the sets of pre-images of Boolean values [8].

Definition 6. Let $p \in P$. The set $p^T = \{d \mid p(d)\downarrow \in T, d \in D\}$ is called the *truth domain* of the predicate p and the set $p^F = \{d \mid p(d) \downarrow \in F, d \in D\}$ is called its *falsity domain*.

The statement given below is obvious.

Lemma 2. Let $p \in P$. Then $p^T \cap p^F = \varnothing$. And vice versa, let $A, B \subseteq D$ and let $A \cap B = \varnothing$. Then there is a unique predicate $p \in P$ for which $p^T = A$ and $p^F = B$.

For tolerance-stable predicates we have the following statement.

Lemma 3. The truth and falsity domains of a tolerance-stable predicate are intolerant sets. And vice versa, let $A, B \subseteq D$ and let $A \not\approx B$. Then there is a unique predicate $p \in PTS$ for which $p^T = A$ and $p^F = B$.

The proof follows immediately from the preceding statements.

4. Infinitary algebra of partial predicates. We use the above notations for definition of operations on the set P. The basic operations over partial predicates are the unary negation operation $\neg : P \xrightarrow{t} P$ and analogues of the Kleene's operations of strong disjunction $\vee$ and strong conjunction $\wedge$.

In contrast to traditional algebras with n-ary operations, the latter operations are set-ary, i.e., their arguments are not n-tuples of predicates but sets of predicates, thus, these operations are of the type $\vee : 2^P \xrightarrow{t} P$ and $\wedge : 2^P \xrightarrow{t} P$.

Definition 7. The infinitary operations of *set-ary disjunction* $\vee : 2^P \xrightarrow{t} P$, *set-ary conjunction* $\wedge : 2^P \xrightarrow{t} P$, and *unary negation* operation $\neg : P \xrightarrow{t} P$ are specified by the following formulas in which $p, p_i \in P\ (i \in I)$:

$$(\vee\{p_i \mid i \in I\})^T = \cup\{p_i{}^T \mid i \in I\},$$

$$(\vee\{p_i \mid i \in I\})^F = \cap\{p_i{}^F \mid i \in I\},$$

$$(\wedge\{p_i \mid i \in I\})^T = \cap\{p_i{}^T \mid i \in I\},$$

$$(\wedge\{p_i \mid i \in I\})^F = \cup\{p_i{}^F \mid i \in I\},$$

$$(\neg p)^T = p^F,\ (\neg p)^F = p^T.$$

It is obvious that the mentioned formulas correctly specify the corresponding operations because truth and falsity domains of specified predicates do not intersect. We note that these definitions can be easily reformulated in conventional functional definition style. For example, for the disjunction operation, we obtain:

$$\vee\{p_i \mid i \in I\}(d) = \begin{cases} T, & \text{if } p_i(d)\downarrow = T \text{ for some } i \in I, \\ F, & \text{if } p_i(d)\downarrow = F \text{ for every } i \in I, \\ \text{underfined} & \text{in other cases.} \end{cases}$$

Definition 8. The algebra $AK(D) = < P, \vee, \wedge, \neg >$ is called the *Kleene infinitary predicate algebra* [9].

We will additionally use the binary disjunction and conjunction operations and also the predicates $\overline{T}$ and $\overline{F}$, treating them as abbreviations of the formulas $p \vee q = \vee\{p, q\}$, $p \wedge q = \wedge\{p, q\}$, $\overline{T} = \wedge\varnothing$, and $\overline{F} = \vee\varnothing$ respectively. We denote the nowhere defined predicate by $\perp$.

Theorem 1. The set of tolerance-stable predicates *PTS* forms a sub-algebra of the algebra *AK*(*D*).

Proof. It is necessary to show the closure of the class *PTS* with respect to the operations of the algebra. We prove this only for the disjunction operation. Let $\{p_i \mid i \in I\} \subseteq PTS$. We should prove that the sets $(\vee\{p_i \mid i \in I\})^T = \cup\{p_i{}^T \mid i \in I\}$ and $(\vee\{p_i \mid i \in I\})^F = \cap\{p_i{}^F \mid i \in I\}$ are intolerant.

Indeed, by the definition of *PTS* for any $p_i \in PTS$ ($i \in I$) we have that $p_i^T \not\approx p_i^F$. This means that $(p_i^T \times p_i^F) \cap \approx = \varnothing$ for any $i \in I$. Hence, $(p_i^T \times \cap\{p_j^F \mid j \in I\}) \cap \approx = \varnothing$ for any $i \in I$.

Therefore $(\cup\{p_j^T \mid j \in I\} \times \cap\{p_j^F \mid j \in I\}) \cap \approx = \varnothing$.

Definition 9. The algebra $ATS(D) =< PTS, \vee, \wedge, \neg >$ is called an *infinitary algebra of tolerance-stable predicates*.

It is precisely the algebra ATS(D) that is the central object being investigated in this article.

5. The system of generators of the algebra of tolerance-stable predicates. To represent classes of predicates, the characteristic predicate χ_d that uniquely determines an element $d \in D$ is commonly used. In terms of truth and falsity domains, this predicate is specified by the formulas $\chi_d^T = \{d\}$ and $\chi_d^F = D \setminus \{d\}$. However, this predicate is not tolerance-stable if there is a d', distinct from *d*, such that $d \approx d'$. Therefore, instead of this characteristic predicate, we will use its best tolerance-stable approximation.

Definition 10. The predicate τ_d specified by formulas $\tau_d^T = \{d\}$ and $\tau_d^F = d^\#$ is called the *tolerance-stable characteristic predicate* of an element $d \in D$.

Having a tolerance-stable characteristic predicate of an element, it is possible to define a tolerance-stable characteristic predicate of a set *A*.

Definition 11. The predicate $TC(A) = \vee\{\tau_a \mid a \in A\}$ is called the *tolerance-stable characteristic predicate of a set* $A \subseteq D$.

This definition is supported by the following formulas:

$$TC(A)^T = (\vee\{\tau_a \mid a \in A\})^T = \cup\{\tau_a^T \mid a \in A\} = = \cup\{\{a\} \mid a \in A\} = A;$$

$$TC(A)^F = (\vee\{\tau_a \mid a \in A\})^F = \cap\{\tau_a^F \mid a \in A\} = \cap\{a^\# \mid a \in A\} = A^\#.$$

Let us also introduce the tolerance-stable characteristic predicate of a pair of sets (*A*, *B*) such that *A* is characterized by the truth domain and *B* is characterized by the falsity domain of the predicate.

Definition 12 Let $A, B \subseteq D$ and let $A \not\approx B$. Then a predicate $NF(A,B) = TC(A) \vee (\neg TC(B) \wedge \perp)$ is called the *tolerance-stable characteristic predicate of a pair* (A, B).

The next lemma justifies the correctness of this definition.

Lemma 4. Let $A, B \subseteq D$ and let $A \not\approx B$. Then $NF(A,B)^T = A$ and $NF(A,B)^F = B$.

Indeed,

$$NF(A,B)^T = (TC(A) \vee (\neg TC(B) \wedge \perp))^T = = (TC(A))^T \cup (\neg TC(B) \wedge \perp)^T = = A \cup ((\neg TC(B))^T \cap \perp^T) = A \cup ((TC(B))^F \cap \varnothing) = A;$$

$$NF(A,B)^F = (TC(A) \vee (\neg TC(B) \wedge \perp))^F = = (TC(A))^F \cap (\neg TC(B) \wedge \perp)^F = = A^\# \cap ((\neg TC(B))^F \cup \perp^F) = = A^\# \cap ((TC(B))^T \cup \varnothing) = A^\# \cap B = A^\#.$$

The last equality follows from the lemma 1.

Now everything is prepared to define any tolerance-stable predicate *p* with the help of tolerance-stable characteristic predicates τ_d and nowhere defined predicate $\perp$. Indeed, *p* is unambiguously characterized by two intolerant sets $A = p^T$ and $B = p^F$, therefore *p* is described by the formula $TC(A) \vee (\neg TC(B) \wedge \perp)$.

This formula is constructed of the symbol of nowhere defined predicate $\perp$ and the symbols of tolerance-stable characteristic predicates τ_d, where $d \in D$. We obtain the following statement.

Theorem 2. The set of the predicates $\{\perp\} \cup \{\tau_d \mid d \in D\}$ is the system of generators of the algebra $ATS(P, D) =< PTS, \vee, \wedge, \neg >$.

6. The system of identical transformations. In the preceding sections we did not make explicit distinction between a formula (a term of the predicate algebra) and the predicate (an element of the algebra), denoted by the formula. There was no need for such a distinction since only properties of predicates in corresponding algebras (semantic aspect) were considered. In constructing of the system of identical transformations (syntactical aspect) such a distinction is necessary to be made.

We assume that the class *TT*(*D*) of formulas (terms of the algebra) is inductively constructed with the help of the symbols of algebraic operations $\vee$, $\wedge$, and $\neg$, the symbol of nowhere defined predicate $\perp$, and the symbols of tolerance-stable characteristic predicates τ_d, where $d \in D$, as follows:

- $\perp \in TT(D)$, $\tau_d \in TT(D)$ for any $d \in D$;
- if $ts \subseteq TT(D)$ and $t \in TT(D)$ then $\vee ts$, $\wedge ts$, and $\neg t$ belong to *TT*(*D*).

We use the *same notation* to denote the symbol of an operation and the operation itself.

The infinitary character of construction of formulas manifests itself only "horizontally," i.e., only by infinity of disjunctions and conjunctions, and the structural complexity of any formula is "vertically" restricted by some finite number.

The interpretation of the formulas in $ATS(D) =< PTS, \vee, \wedge, \neg >$ is defined in the usual way. A formal identity is an expression of the form $t = t'$, where $t, t' \in TT(D)$. If an identity $t = t'$ is true in the algebra $ATS(D)$, we write $ATS(D) \models t = t'$. Identities are treated as axiom schemes in equational logics [9]. If an equality $t = t'$ is inferred from the system of identities *E*, we write $E \vdash t = t'$.

Let us go now to construction of the complete set of identities in the algebra. This system is based on the fact that the set *PTS* with operations $\vee$ and $\wedge$ is a complete distributive lattice with an involution $\neg$ (axioms E1–E6) [9], for which additional axioms E7–E10 define special properties of tolerance-stable characteristic predicates.

Thus, we obtain the following system of identities, in which X, X', X_i, and X_{ij} denote arbitrary terms from *TT*(*D*), $a, b \in D$, $A \subseteq D$, and which therefore are *axiom schemes* of identities (in a number of axioms we use binary connectives $\vee$ and $\wedge$):

E1: $\vee\{\vee\{X_{ij} \mid j \in J_i\} \mid i \in I\} = \vee\{X_{ij} \mid j \in J_i, i \in I\}$ (associativity);

E2: $\vee\{\wedge\{X_{ij} \mid j \in J_i\} \mid i \in I\} = \wedge\{\vee\{X_{i\pi(i)} \mid i \in I\} \mid \pi \in \Pi_{i\in I} J_i\}$ (complete distributivity);

E3: $\vee\{X, \vee\{X, X'\}\} = X$ (absorption);

E4: $\neg\neg X = X$ (involution);

E5: $\neg \vee \{\{X_i \mid i \in I\} = \wedge\{\neg X_i \mid i \in I\}$ (the De Morgan rule);

E6: $\neg \bot = \bot$ ($\bot$ – fixed point under involution);

E7: $\tau_a \wedge \neg \tau_b = \tau_a$ if $a \not\approx b$ (conjunctive reduction);

E8: $\tau_a = \tau_a \vee (\neg \vee \{\tau_b \mid b \in a^{\#}\}) \wedge \bot$ (normalization);

E9: $\wedge\{\tau_a \mid a \in A\} \vee \bot = \bot$ for any $A \subseteq D$ such that cardinality of *A* is greater than 1 (contrariety of tolerance-stable characteristic predicates);

E10: $\wedge\varnothing = \vee\{\tau_d \mid d \in D\}$ (completeness of the class of tolerance-stable characteristic predicates).

We treat a system of identities as a special infinitary equational calculus [9].

Lemma 5. The inference relation based on the system of identities E1– E10 is sound in the algebra $ATS(D)$, i.e.

$$\text{E1–E10} \vdash t = t' \Rightarrow ATS(D) \models t = t' \text{ for any } t, t' \in TT(D).$$

Proof. The proof of the lemma consists of two parts. In the first part, the soundness of identities E1–E10 is proved and, in the second part, preservation of the soundness under inference rules of equational logic is proved. Since the proof of the second part is traditional for equational logic, it is not considered here. The proof of soundness of identities is based on the check of the coincidence of the truth and falsity domains of the left-hand and right-hand sides of the identities. Here we prove only the soundness of E7–E10.

Let us show the coincidence of the truth domains of the left and right-hand sides of E7. Indeed, since $a \not\approx b$:

$$(\tau_a \wedge \neg \tau_b)^T = \tau_a^T \cap (\neg \tau_b)^T = \{a\} \cap \tau_b^F = \{a\} \cap b^{\#} = \{a\} = \tau_a^T.$$

Now we show the coincidence of the falsity domains:

$$(\tau_a \wedge \neg \tau_b)^F = \tau_a^F \cup (\neg \tau_b)^F = a^{\#} \cup \tau_b^T = a^{\#} \cup \{b\} = a^{\#} = \tau_a^F.$$

Then we prove the soundness of E8. At first, we evaluate the truth domains:

$$(\tau_a \vee (\neg \vee \{\tau_b \mid b \in a^{\#}\}) \wedge \bot)^T = \tau_a^T \cup (\neg \vee \{\tau_b \mid b \in a^{\#}\})^T \cap \bot^T = \tau_a^T \cup (\neg \vee \{\tau_b \mid b \in a^{\#}\})^T \cap \varnothing = \tau_a^T.$$

Now for the falsity domains of E8 we have:

$$(\tau_a \vee (\neg \vee \{\tau_b \mid b \in a^{\#}\}) \wedge \bot)^F = \tau_a^F \cap (\neg \vee \{\tau_b \mid b \in a^{\#}\})^F \cup \bot^F = a^{\#} \cap (\vee\{\tau_b \mid b \in a^{\#}\})^T \cup \varnothing = a^{\#} \cap (\cup\{\tau_b^T \mid b \in a^{\#}\}) = a^{\#} \cap (\cup\{\{b\} \mid b \in a^{\#}\}) = a^{\#} \cap a^{\#} = a^{\#}.$$

In the same way we prove the soundness of E9. Since $a \neq b$ we have

$$(\tau_a \wedge \tau_b \vee \bot)^T = (\tau_a^T \cap \tau_b^T) \cup \bot^T = \{a\} \cap \{b\} \cup \varnothing = \varnothing = \bot^T$$

and

$$(\tau_a \wedge \tau_b \vee \bot)^F = (\tau_a^F \cup \tau_b^F) \cap \bot^T = (\tau_a^F \cup \tau_b^F) \cap \varnothing = \varnothing = \bot^F.$$

At last, we prove the soundness of E10. Since $(\wedge\varnothing)^T = D$ and $(\wedge\varnothing)^F = \varnothing$ we have

$$(\vee\{\tau_d \mid d \in D\})^T = \cup\{\tau_d^T \mid d \in D\} = \cup\{d \mid d \in D\} = D$$

and

$$(\vee\{\tau_d \mid d \in D\})^F = \cap\{\tau_d^F \mid d \in D\} = \cap\{d^{\#} \mid d \in D\} = \varnothing.$$

The lemma is proved.

Finally, we prove the completeness of the system of schemes E1–E10. Such a proof is usually based on the reduction of any formula to its normal form. In our case, a normal form is of the form *NF*(*A*, *B*).

Definition 13. The formulas (terms) of the set

$$TNF(D) = \{NF(A, B) \mid A, B \subseteq D, A \not\approx B\}$$

are called the *positive-negative normal forms*.

In what follows, we simply call such forms *normal*. They are similar to the forms defined in [8]. We now prove the basic property of such normal forms.

Theorem 3. The normal forms from the set *TNF*(*D*) bijectively specify the predicates of the set *PTS*.

In fact, as has been shown earlier, any predicate *p* from *PTS* can be represented by the formula $NF(p^T, p^F)$. Therefore, it remains to show the unambiguity of such a representation. Let $NF(A_1, B_1)$, $NF(A_2, B_2) \in TNF(D)$, and let $NF(A_1, B_1) \neq NF(A_2, B_2)$. By the construction of formulas, this means that $A_1 \neq A_2$ or $B_1 \neq B_2$. But then, in each case, due to lemma 3, the predicates specified by these formulas are also different.

Preparatory to proving the completeness, we will show for an arbitrary term $t \in TT(D)$ the inferrability of its normal form denoted by *nf*(*t*).

Lemma 6. E1–E10 $\vdash t = nf(t)$ for any $t \in TT(D)$.

Proof. We prove the lemma by induction on the construction of *t*. The scrupulous presentation of all the details of the inference of a normal form is very cumbersome; therefore, we will restrict ourselves to only the specification of basic stages of transformation of an arbitrary term into its normal form. First, we note that, for the identities of schemes E1–E6, the corresponding dual identities can be inferred [8]. Moreover, the axiom of idempotency $\vee\{X\} = X$ is inferred. In fact, from the axiom of absorption E3, we obtain the identity $X = \vee\{X, \wedge\{X, \vee\{X\}\}\}$ after replacing X' by $\vee\{X\}$. Then, in the axiom dual to the axiom of absorption, we replace X' by *X* obtaining the identity $X = \wedge\{X, \vee\{X\}\}$. Using the identities obtained, we infer the axiom of idempotency $\vee\{X\} = X$.

To prove the lemma we should consider all five cases of construction of the term *t*.

1. Let $t = \perp$. In this case $nf(t) = (\vee\varnothing) \vee (\neg(\vee\varnothing) \wedge \perp)$.

We conduct the proof by transforming first the formula *nf* (*t*) into the simpler formula $\perp$. Based on the reducibility to $\perp$ and taking into account that all the identical transformations are invertible, we will be able to reduce $\perp$ to *nf* (*t*).

We have the following sequence of transformations:

$$nf(t) = (\vee\varnothing) \vee (\neg(\vee\varnothing) \wedge \perp) = (\vee\varnothing) \vee ((\wedge\varnothing) \wedge (\wedge\{\perp\})) =$$
$$= (\vee\varnothing) \vee (\wedge(\varnothing \cup \{\perp\})) = (\vee\varnothing) \vee (\wedge\{\perp\}) = (\vee\varnothing) \vee \perp =$$
$$= (\vee\varnothing) \vee (\vee\{\perp\}) = \vee(\varnothing \cup \{\perp\}) = \vee\{\perp\} = \perp .$$

2. Let $t = \tau_d$ for an arbitrary $d \in D$. The reducibility of *t* to its normal form *nf* (*t*) is specified by the axiom of normalization E8.

3. Let $t = \vee\{t_i \mid i \in I\}$.

We should consider three cases:

1) the cardinality of *I* is equal to 0;
2) the cardinality of *I* is equal to 1;
3) the cardinality of *I* is greater than 1.

In the first case (when the cardinality of *I* is equal to 0) the term *t* is $\vee\varnothing$. Its normal form is $nf(\vee\varnothing) = (\vee\varnothing) \vee (\neg(\vee\{\tau_d \mid d \in D\}) \wedge \perp)$. Using E10, E5, and E3 we obtain the following transformation:

$$(\vee\varnothing) \vee (\neg(\vee\{\tau_d \mid d \in D\}) \wedge \perp) = (\vee\varnothing) \vee (\neg \wedge \varnothing) \wedge \perp =$$
$$= (\vee\varnothing) \vee (\vee\varnothing) \wedge \perp = \vee\varnothing .$$

In the second case (when the cardinality of *I* is equal to 1) the term *t* has the form $\vee\{t'\}$. Using the idempotency axiom we prove this case.

The third case (when the cardinality of *I* is greater than 1) is more difficult.

By the induction hypothesis, $E1–E9 \vdash t_i = nf(t_i)$ for all $i \in I$.

According to the inference rules of equational logic, the equality $\vee\{t_i \mid i \in I\} = \vee\{nf(t_i) \mid i \in I\}$ is inferred. Therefore, it remains to show the inference of the equality $\vee\{nf(t_i) \mid i \in I\} = nf(\vee\{nf(t_i) \mid i \in I\})$.

Let

$nf(t_i) = (\vee\{\tau_a \mid a \in A_i\}) \vee (\neg \vee \{\tau_b \mid b \in B_i\} \wedge \perp), i \in I$.

Then the normal form of $t = \vee\{t_i \mid i \in I\}$ is $NF(\cup\{A_i \mid i \in I\}, \cap\{B_i \mid i \in I\})$. Now we show that the required equality can be inferred. We have the following sequence of transformations:

$$\vee\{nf(t_i) \mid i \in I\} = \vee\{(\vee\{\tau_a \mid a \in A_i\}) \vee (\neg \vee \{\tau_b \mid b \in B_i\} \wedge \perp) \mid i \in I\} =$$
$$= (\vee\{\vee\{\tau_a \mid a \in A_i\} \mid i \in I\}) \vee (\vee\{(\neg \vee \{\tau_b \mid b \in B_i\} \wedge \perp) \mid i \in I\}) =$$
$$= (\vee\{\tau_a \mid a \in A_i, i \in I\}) \vee \neg(\wedge\{(\vee\{\tau_b \mid b \in B_i\} \vee \neg \perp) \mid i \in I\}) =$$
$$= (\vee\{\tau_a \mid a \in \cup\{A_i \mid i \in I\}\}) \vee \neg(\wedge\{\vee(\{\tau_b \mid b \in B_i\} \cup \{\perp\}) \mid i \in I\})$$

.

The first part of the formula obtained is $TC(\cup\{A_i \mid i \in I\})$. Now we transform the formula $\wedge\{\vee(\{\tau_b \mid b \in B_i\} \cup \{\perp\}) \mid i \in I\}$.

Using the axiom of distributivity E2, we obtain conjuncts of the following three forms:

(1) consisting of symbols of tolerance-stable characteristic predicates only;

(2) consisting of symbols of tolerance-stable characteristic predicates and the symbol of the nowhere defined predicate $\perp$;

(3) equal to $\perp$.

By the axiom of absorption, all the conjuncts of the second type are reduced to the symbol $\perp$. Therefore, we have

$$\wedge\{\vee(\{\tau_b \mid b \in B_i\} \cup \{\perp\}) \mid i \in I\} =$$
$$= \vee\{\wedge\{\tau_{\pi(i)} \mid i \in I\} \mid \pi \in \Pi\{B_i \mid i \in I\}\} \vee \perp .$$

Then (after trivial transformations) we use E9:

$$\vee(\{\wedge\{\tau_{\pi(i)} \mid i \in I\} \mid \pi \in \Pi\{B_i \mid i \in I\}\} \vee \perp) =$$
$$= \vee\{\wedge\{\perp\}\} = \vee\{\perp\} = \perp .$$

Thus, we proved the lemma for the case $t = \vee\{t_i \mid i \in I\}$.

4. Let $t = \wedge\{t_i \mid i \in I\}$. The proof is similar to the preceding one.

5. Let $t = \neg t'$. By the induction hypothesis, $E1–E10 \vdash t' = nf(t')$. Assume that $nf(t') = TC(A) \vee \neg TC(B) \wedge \perp$. Then the normal form of *t* is $nf(t) = TC(B) \vee \neg TC(A) \wedge \perp$.

Let us make the following transformations:

$$\neg nf(t') = \neg(TC(A) \vee \neg TC(B) \wedge \perp) =$$
$$= \neg TC(A) \wedge \neg(\neg TC(B) \wedge \perp) =$$
$$= \neg TC(A) \wedge (TC(B) \vee \perp) = (\neg TC(A) \wedge$$
$$\wedge TC(B)) \vee (\neg TC(A) \wedge \perp).$$

Transforming the formula $\neg TC(A) \wedge TC(B)$, we obtain

$$\neg TC(A) \wedge TC(B) = \neg \vee \{\tau_a \mid a \in A\} \wedge (\vee\{\tau_b \mid b \in B\}) =$$
$$= \wedge\{\neg\tau_a \mid a \in A\} \wedge (\vee\{\tau_b \mid b \in B\}) =$$
$$= \vee\{\tau_b \wedge (\wedge\{\neg\tau_a \mid a \in A\}) \mid b \in B\} =$$
$$= \vee\{\wedge\{\tau_b \wedge \neg\tau_a \mid a \in A\} \mid b \in B\} =$$
$$= \vee\{\wedge\{\tau_b \mid a \in A\} \mid b \in B\} =$$
$$= \vee\{\tau_b \mid b \in B\} = TC(B) .$$

The lemma is proved.

Let us prove the completeness of the inference relation.

Lemma 7. The inference relation based on the system of identities E1– E10 is complete in the algebra ATS(D) , i.e.

$$\text{E1–E10} \models t = t' \Rightarrow \text{ATS(D)} \vdash t = t'$$

for any $t, t' \in TT(D)$.

Proof. The completeness is proved by reducing left- and right-sides of the equation to their normal form. If $\models t = t'$, then we have *nf* (*t*) = *nf* (t′). Since, according to the preceding statement, E1–E10 $\vdash t = nf(t)$ and E1–E10 $\vdash t' = nf(t')$, we have E1–E10 $\vdash t = t'$.

Theorem 4. The system of identities E1–E10 is sound and complete in the algebra ATS(D), i.e.

$$\text{E1–E10} \vdash t = t' \Leftrightarrow \text{ATS(D)} \models t = t' \text{ for any } t, t' \in TT(D).$$

7. Conclusions. Fundamental binary relations such as equivalence and partial order are intensively studied and used in various fields of science. Tolerance relations still lack careful investigations though they are more and more actively used in various branches of informatics.

In the paper we have studied special predicates defined on a set with a tolerance relation. Such predicates (called tolerance-stable), if defined on tolerant elements, should yield on them the same Boolean values. We have defined algebras of tolerance-stable predicates and construct a system of identical transformations which is sound and complete. It means that an infinitary equational logic of tolerance-stable predicates has been constructed.

In the forthcoming papers we plan to define classes of programs stable under tolerance relations, construct corresponding program algebras, and specify a system of identical program transformation.

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