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THE PROBLEM OF MEETING OF N FUZZY OBJECTS

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In the paper a problem of meeting of fuzzy linear objects (fuzzy linear differential inclusions) is considered and the necessary condition of optimality is obtained.

KEY WORDS: fuzzy objects, meeting problem, fuzzy linear differential inclusions, necessary condition of optimality.

ЗАДАЧА ВСТРЕЧИ N НЕЧЕТКИХ ОБЪЕКТОВ

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В работе рассматривается задача встречи нечетких линейных объектов (нечетких линейных дифференциальных включений) и получено необходимое условие оптимальности.

КЛЮЧЕВЫЕ СЛОВА: нечеткие объекты, проблема встречи, нечеткие линейные дифференциальные включения, необходимое условие оптимальности.

ПРОБЛЕМА ЗУСТРІЧІ N НЕЧІТКИХ ОБ’ЄКТІВ

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В роботі розглядається проблема зустрічі нечітких лінійних об’єктів (нечітких лінійних диференціальних включень) і отримана необхідна умова оптимальності.

КЛЮЧОВІ СЛОВА: нечітки об’єкти, проблема зустрічі, нечіткі лінійні диференціальні включення, необхідна умова оптимальності.

1. Introduction. The first research of the differential equations with set-valued right-hand side was fulfilled by A. Marchaud and S.C. Zaremba. In the early sixties, T. Wazewski, A.F. Filippov obtained fundamental results about existence and properties of solutions of the differential equations with set-valued right-hand side (differential inclusions). Connection between differential inclusions and optimum control problems was one of the most important results of these papers. These results made an impulse for development of the theory of differential inclusions [1–3]. In work [4] the notion of R-solution for differential inclusion was introduced as an absolutely continuous set-valued map. Various problems for the R-solution theory were considered in [2,5]. The basic idea for a development of an equation for R-solutions (integral funnels) contains in [6].

In the eighty years of the last century the control theory in the conditions of uncertainty began to be formed. The control differential equations with set of initial conditions [7–9], control set differential equations [10–12] and the control differential inclusions [13–16] are used in the given theory for exposition of dynamic processes.

In recent years, the fuzzy set theory introduced by L.A. Zadeh [17] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations [18–21] and inclusions [20,22–24] as well as in the theory of control fuzzy differential equations [25,26] and inclusions [27–29].

In this paper we consider a problem of a meeting of fuzzy linear objects and we receive a necessary condition of optimality.

2. Main definitions. Let $\mathrm{comp}(R^n)(\mathrm{conv}(R^n))$ be a set of all nonempty compact (and convex) subsets from the space R^n, $h(A,B)=\min\limits_{r\ge 0}\{S_r(A)\supset B,\, S_r(B)\supset A\}$ be the Hausdorff distance between sets A and B, $S_r(A)$ be r-neighborhood of set A.

Let E^n be the set of all maps $u: R^n\to[0,1]$ such that u satisfies the following conditions:

1) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0)=1$;
2) u is fuzzy convex, that is, $u(\lambda x+(1-\lambda)y) \ge \min\{u(x),u(y)\}$ for any $x,y \in R^n$ and $0 \le \lambda \le 1$;
3) u is upper semicontinuous,
4) $[u]^0 = cl\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0 < \alpha \le 1$, denote $[u]^\alpha = \{x \in R^n : u(x) \ge \alpha\}$.

Then from 1)–4), it follows that the α-level set $[u]^\alpha \in conv(R^n)$ for all $0 \le \alpha \le 1$.

Let θ be the fuzzy mapping defined by $\theta(x)=0$ if $x \ne 0$ and $\theta(0)=1$.

Define $D: E^n \times E^n \to [0,\infty)$ by the relation $D(u,v) = \sup\limits_{0\le\alpha\le1} h([u]^\alpha,[v]^\alpha)$. Then D is a metric in E^n.

Further we know that [30]:

- (E^n, D) is a complete metric space,
- $D(u+w,v+w) = D(u,v)$ for all $u,v,w \in E^n$,
- $D(\lambda u,\lambda v) = |\lambda| D(u,v)$ for all $u,v \in E^n$ and $\lambda \in R$.

Definition 1. [18] A mapping $F:[0,T] \to E^n$ is measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : [0,T] \to conv\left(R^n\right)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

Definition 2. [18] A mapping $F:[0,T] \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \le h(t)$ for every $x(t) \in F_0(t)$.

Definition 3. [18] The integral of a fuzzy mapping $F:[0,T] \to E^n$ is defined levelwise by $\left[\int_0^T F(t)dt\right]^\alpha = \int_0^T F_\alpha(t)dt$. The set $\int_0^T F_\alpha(t)dt$ of all $\int_0^T f(t)dt$ such that $f:[0,T] \to R^n$ is a measurable selection for $F_\alpha : [0,T] \to conv(R^n)$ for all $\alpha \in [0,1]$.

Definition 4. [18] A measurable and integrably bounded mapping $F:[0,T] \to E^n$ is said to be integrable over $[0,T]$ if $\int_0^T F(t)dt \in E^n$.

Now we consider the following controlled differential equations with the fuzzy parameter

$$\dot{x} = f(t,x,w,v),\ x(0)=x_0, \tag{1}$$

where $\dot{x}$ denotes dx/dt; $t \in R_+$ is the time; $x \in R^n$ is the state; $w \in R^m$ is the control; $v \in V \in E^k$ is the fuzzy parameter; $f: R_+ \times R^n \times R^m \times R^k \to R^n$.

Let $W: R_+ \to conv(R^m)$ be the measurable set-valued map.

Definition 5. The set LW of all measurable single-valued branches of the set-valued map $W(\cdot)$ is the set of the admissible controls.

Further consider the following controlled fuzzy differential inclusions

$$\dot{x} \in F(t,x,w),\quad x(0)=x_0, \tag{2}$$

where $F: R_+ \times R^n \times R^m \to E^n$ is the fuzzy map such that $F(t,x,w) \equiv f(t,x,w,V)$.

Obviously, controlled fuzzy differential inclusion (2) turns into the ordinary fuzzy differential inclusion

$$\dot{x} \in \Phi(t,x),\ x(0)=x_0, \tag{3}$$

if the control $\tilde{w}(\cdot) \in LW$ is fixed and $\Phi(t,x) \equiv F(t,x,\tilde{w}(t))$.

If right-hand side of the fuzzy differential inclusion (3) satisfies some conditions (for example see [24]) then fuzzy differential inclusions (3) has the fuzzy R-solution.

Let $X(t)$ denotes fuzzy R-solution of the differential inclusion (3), then $X(t,w)$ denotes the fuzzy R-solution of controlled differential inclusion (2) for the fixed $w(\cdot) \in LW$.

Definition 6. The set $Y(T) = \{X(T,w) : w(\cdot) \in LW\}$ is called the attainable set of the fuzzy system (2).

3. Some properties of the fuzzy R-solution. Further in the given paper, we consider the following controlled linear fuzzy differential inclusions

$$\dot{x} \in A(t)x + G(t,w),\ x(0)=x_0, \tag{4}$$

where $A(t)$ is $(n\times n)$-dimensional matrix-valued function; $G: R_+ \times R^m \to E^n$ is a fuzzy map.

In this section, we consider some properties of the fuzzy R-solution of controlled fuzzy differential inclusion (4).

Let the following condition be true:

Condition A:

1. $A(\cdot)$ is measurable on $[0,T]$;
2. The norm $\|A(t)\|$ of the matrix $A(t)$ is integrable on $[0,T]$;
3. The set-valued map $W:[t_0,T] \to conv\left(R^m\right)$ is measurable on $[0,T]$;

4. The fuzzy map $G:[0,T]\times R^m \to E^n$ satisfies the conditions
   a) measurable in t;
   b) continuous in w;

5. There exist $v(\cdot)\in L_2[0,T]$ and $l(\cdot)\in L_2[0,T]$ such that

$$h(W(t),0)\le v(t),\ D(G(t,w),\theta)\le l(t)$$

almost everywhere on $[0,T]$ and all $w\in W(t)$.

6. The set $Q(t)=\{G(t,w(t)) : w(\cdot)\in LW\}$ is compact and convex for almost every $t\in[0,T]$, i.e. $Q(t)\in conv(E^n)$.

Theorem 1 [29]. *Let the condition A be true. Then for every $w(\cdot)\in LW$ there exists the fuzzy R-solution $X(\cdot,w)$ such that*

1) the fuzzy map $X(\cdot,w)$ is equal to

$$X(t,w)=\Phi(t)x_0+\Phi(t)\int_0^t \Phi^{-1}(s)G(s,w(s))ds,$$

where $t\in[0,T]$; $\Phi(t)$ is a Cauchy matrix of the differential equation $\dot{x}=A(t)x$;

2) $X(t,w)\in E^n$ for every $t\in[0,T]$;

3) the fuzzy map $X(\cdot,w)$ is the absolutely continuous fuzzy map on $[0,T]$.

Theorem 2 [29]. *Let the condition A be true. Then the attainable set $Y(T)$ is compact and convex.*

4. The problem of meeting of N fuzzy objects. Consider N linear controlled differential inclusions with fuzzy parameters

$$\dot{x}^i\in A^i(t)x^i+G^i(t,w^i),\ x^i(0)=x_0^i,\ i=\overline{1,N},\quad (5)$$

where $x^i\in R^n$; $t\in R_+$; $A^i(t):R_+\to R^{n\times n}$ is a matrix $n\times n$; $G^i(t,w^i):R_+\times R^{k_i}\to E^n$ is a fuzzy map; $w^i\in W^i\subset R^{k_i}$ is a control parameter; $x_0^i\in R^n$.

Consider the following optimal control problem (*problem A*): it is necessary to find the minimal time T^* and controls $w_*^i(\cdot)\in LW^i$, $i=\overline{1,N}$ such that the fuzzy R-solutions of system (5) satisfy of the condition:

$$\bigcap_{i=1}^{N} X^i(T^*,w_*^i)\ne\varnothing.\quad (6)$$

Definition 7. The set $(T^*,w_*^1(\cdot),...,w_*^N(\cdot))$ is said to be optimal if

$$\bigcap_{i=1}^{N} X^i(T^*,w_*^i)\ne\varnothing \text{ and } \bigcap_{i=1}^{N} X^i(\tau,w^i)=\varnothing$$

for every $0\le\tau<T^*$ and all $w^i(\cdot)\in LW^i$, $i=\overline{1,N}$.

Let us reduce the necessary conditions of optimality of set $(T^*,w_*^1(\cdot),...,w_*^N(\cdot))$ for meeting problem.

Theorem 3. *Let the following conditions hold for every $i\in\{1,...,N\}$:*

1) $A^i(\cdot)$ is measurable on $[0,T^*]$;

2) The norm $\|A^i(t)\|$ of the matrix $A^i(t)$ is integrable on $[0,T^*]$;

3) The set-valued map $W^i:[t_0,T]\to conv(R^{k_i})$ is measurable on $[0,T^*]$;

4) The fuzzy map $G^i:[0,T^*]\times R^{k_i}\to E^n$ satisfies the conditions
   a) measurable in t;
   b) continuous in w^i;

5) There exist $v^i(\cdot)\in L_2[0,T^*]$ and $l^i(\cdot)\in L_2[0,T^*]$ such that

$$h(W^i(t),0)\le v^i(t),\ D(G^i(t,w^i),\theta)\le l^i(t)$$

almost everywhere on $[0,T^*]$ and all $w^i\in W^i(t)$.

6) The set $Q^i(t)=\{G^i(t,w^i(t)) : w^i(\cdot)\in LW^i\}$ is compact and convex for almost every $t\in[0,T^*]$, i.e. $Q^i(t)\in conv(E^n)$ and the set $(T^*,w_*^1(\cdot),...,w_*^N(\cdot))$ is optimal for the problem (5),(6).

Then there exist $j\in\{1,...,N\}$ and solution $\psi^j(\cdot)$ of the differential equation $\dot{\psi}^j=-(A^j(t))^T\psi^j$, $\|\psi^j(T^*)\|=1$ such that

1)
$$C([G^j(t,w_*^j)]^1,\psi^j(t))=\max_{w^j\in W^j(t)} C([G^j(t,w^j)]^1,\psi^j(t))$$

almost everywhere on $[0,T^*]$;

2)
$$C\left(\left[X^j(T^*,w_*^j)\right]^1,\psi^j(T^*)\right)=-C\left(\left[\bigcap_{i=1}^{N} X^i(T^*,w_*^i)\right]^1,-\psi^j(T^*)\right).$$

Proof. Let us associate with controlled fuzzy system (5) the following controlled fuzzy system

$$\dot{x}\in A(t)x+G(t,w),\ x(0)=x_0,\quad (7)$$

where $x=\left(x^1,...,x^N\right)$, $x^i\in R^n$, $i=\overline{1,N}$, $w=\left(w^1,...,w^N\right)$, $w^i\in R^{k_i}$, $i=\overline{1,N}$,

$$A(t)=\begin{pmatrix} A^1(t) & 0 & ... & 0 \\ 0 & A^2(t) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & A^N(t) \end{pmatrix},$$

$$G(t,w)=\begin{pmatrix} G^1\left(t,C^1w\right) \\ G^2\left(t,C^2w\right) \\ \vdots \\ G^N\left(t,C^Nw\right) \end{pmatrix},$$

$$C^1=\begin{pmatrix} I^1 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}, \ldots,$$

$$C^N=\begin{pmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & I^N \end{pmatrix},$$

I^i is a unit matrix $\left(k_i\times k_i\right)$, $W=\prod_{i=1}^{N}W^i$, $x_0=\left(x_0^1,...,x_0^N\right)^T$.

Under the conditions of theorem, we have conditions

1. $A(\cdot)$ is measurable on $[0,T^*]$;
2. The norm $\left\|A(t)\right\|$ of the matrix $A(t)$ is integrable on $[0,T^*]$;
3. The set-valued map $W:[0,T^*]\to conv\left(R^{k_1\times ..\times k_N}\right)$ is measurable on $[0,T^*]$;
4. The fuzzy map $G:[0,T]\times R^{k_1\times ..\times k_N}\to E^{Nn}$ satisfies the conditions
   a) measurable in t;
   b) continuous in w;
5. There exist $v(t)=\sup_{i=\overline{1,N}} v^i(t)$ and $l(t)=\sup_{i=\overline{1,N}} l^i(t)$ such that

$$h(W(t),0)\le v(t),\ D(G(t,w),\theta)\le l(t)$$

almost everywhere on $[0,T^*]$ and all $w\in W(t)$.

6. The set $Q(t)=\left\{G(t,w(t)):w(\cdot)\in LW\right\}$ is compact and convex for almost every $t\in[0,T^*]$, i.e. $Q(t)\in conv(E^{Nn})$.

Denote by $S_K\in E^{Nn}$ the fuzzy set such that $\left[S_K\right]^\alpha=\left\{x\in R^{Nn} \mid x^1=...=x^N,\ x^i\in R^n, i=\overline{1,N}\right\}$ for all $\alpha\in[0,1]$.

Consider the following optimal controlled problem (*problem B*): it is necessary to find the minimal time T^* and the control $w^*\in LW$ such that the fuzzy R-solution of system (7) satisfies the condition $X\left(T^*,w^*\right)\cap S_K\ne\varnothing$.

Using the results of [16], we know that *problem A* and *problem B* are the equivalent, i.e. the set $\left(T^*,w_*^1(\cdot),...,w_*^N(\cdot)\right)$ is optimal for *problem A* if and only if the set $\left(T^*,w^*(\cdot)\right)$ is optimal for *problem B*, where $w^*(\cdot)=\left(w_*^1(\cdot),...,w_*^N(\cdot)\right)$.

Hence and by [29], it follows that there exists a solution $\psi(\cdot)$ of the differential equation $\dot\psi=-A^T(t)\psi$, $\psi\left(T^*\right)\in S_1(0)$ such that

1) $C([G(t,w^*)]^1,\psi(t))=\max_{w\in W(t)} C(\left[G(t,w)\right]^1,\psi(t))$

almost everywhere on $[0,T^*]$;

2) $C\left(\left[X(T^*,w^*)\right]^1,\psi(T^*)\right)=-C\left(\left[S_k\right]^1,-\psi(T^*)\right)$.

From 1), 2) the statements of the theorem follow. The theorem is proved.

5. Conclusion. Obviously it is possible to consider the other problem of an $\alpha-$meeting: it is necessary to find the minimal time T_α^* and the controls $w_\alpha^i(\cdot)\in LW^i$, $i=\overline{1,N}$ such that the fuzzy R-solutions of system (6) satisfy the condition $\bigcap_{i=1}^{N}\left[X^i\left(T_\alpha,w_\alpha^i\right)\right]^\alpha\ne\varnothing$.

In this case, necessary conditions will be the following: there exist $j\in\{1,...,N\}$ and a solution $\psi^j(\cdot)$ of the differential equation $\dot\psi^j=-\left(A^j(t)\right)^T\psi^j$, $\left\|\psi^j\left(T_\alpha^*\right)\right\|=1$ such that

$$1)\quad \begin{aligned} &C([G^j(t,w_\alpha^j)]^\alpha,\psi^j(t))= \\ &=\max_{w^j\in W^j(t)} C(\left[G^j(t,w^j)\right]^\alpha,\psi^j(t)) \end{aligned}$$

almost everywhere on $[0,T_\alpha^*]$;

2) $$C\left(\left[X^j(T^*_\alpha, w^j_\alpha)\right]^\alpha, \psi^j(T^*_\alpha)\right) = -C\left(\left[\bigcap_{i=1}^{N} X^i\left(T^*_\alpha, w^i_\alpha\right)\right]^\alpha, -\psi^j(T^*_\alpha)\right).$$

Finally, note that $T^*_0 \le T^*_\alpha \le T^*_1 = T^*$ as $\left[X^i\left(t, w^i\right)\right]^0 \subset \left[X^i\left(t, w^i\right)\right]^\alpha \subset \left[X^i\left(t, w^i\right)\right]^1$ for every $t \ge 0$ and $w^i(\cdot) \in LW^i$, $i = \overline{1, N}$.

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