UDC 30A50

MORERA TYPE THEOREM FOR A FAMILY OF CIRCULAR SECTORS

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The exact value for the smallest radius of the ball, in which the given family of sets has the Morera property, is obtained in the paper. The family consists of the circular sectors. The set family with the Morera property in a ball of radius smaller than the minimum of the same radii for each set is constructed in the paper.

KEY WORDS: Morera property, family of circular sectors, Morera type theorem.

ТЕОРЕМА ТИПА МОРЕРЫ ДЛЯ СЕМЕЙСТВА КРУГОВЫХ СЕКТОРОВ

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В работе получено точное значение для наименьшего радиуса шара, в котором заданное семейство множеств обладает свойствами Мореры. Семейство состоит из круговых секторов. В работе сконструировано семейство множеств, обладающее свойствами Мореры, в шаре с радиусом меньшим, чем минимальное значение этого же радиуса для каждого множества.

КЛЮЧЕВЫЕ СЛОВА: свойства Мореры, семейство круговых секторов, теорема типа Мореры.

ТЕОРЕМА ТИПУ МОРЕРИ ДЛЯ СІМ'Ї ШАРОВИХ СЕКТОРІВ

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В роботі отримано точне значення найменьшого радіусу кулі, в якій задана сім'я множин, що мають властивості Морери. Сім'я складається з шарових секторів. В роботі сконструйована сім'я множин, які мають властивості Морери, в кулі, радіус якої меньший, ніж мінімальне значення того ж радіусу для каждої множини.

КЛЮЧОВІ СЛОВА: властивості Морери, сім'я шарових секторів, теорема типу Морери.

We shall use the standard notation Z, R and C for the ring of integers, and the fields of real, and complex numbers, respectively. We also set $Z_+ = \{n \in Z : n \geq 0\}$. Let R^n be the real Euclidean space of dimension $n \geq 2$ with Euclidean norm $|\cdot|$. We shall use the notation $M(n)$ for the group of Euclidean motions in R^n. For a non-empty subsets A, B in R^n, we put $\mathrm{Mot}(A,B) = \{\lambda \in M(n) : \lambda A \subset B\}$.

If A is a non-empty subset in R^n, then ∂A is the boundary of A, $\overline{A}$ is the closure of A. For a non-empty compact subset A in R^n and a real positive number μ, we denote $\mu A = \{x \in R^n : x/\mu \in A\}$, and let $r*(A)$ be the radius of the smallest closed ball containing the set A.

Let B be a non-empty open subset in R^2. Denote by $L_{loc}(B)$ the collection of all functions $f : B \to C$ such that $f\big|_A \in L_1(A)$ for each compact subset . For $m \in Z_+$, denote by $C^m(B)$ the collection of all functions $f : B \to C$ such that $\partial^\alpha f$ is continuous function in B for each $\alpha = (\alpha_1, \alpha_2)$: $\alpha_{1,2} \in Z_+$, $\alpha_1 + \alpha_2 \leq m$. Also let $C^\infty(B) = \bigcap_{m=0}^{\infty} C^m(B)$.

Throughout the point $(x; y) \in R^2$ is identified with the complex number $z = x + iy = \rho e^{i\phi}$ ($\rho = |z|$, $\phi = \arg z \in (-\pi; \pi]$). Then the group of Euclidean motions of the complex plane can be identified with $M(2)$. Let B be a domain in C and let $\mathrm{Hol}(B)$ be the following set of functions from B into C: $f \in \mathrm{Hol}(B)$ if and only if there exists a holomorphic function in B coinciding with f almost everywhere (with respect to the Lebesgue measure).

Suppose that A is a Jordan domain in C with rectifiable boundary ∂A. We will say that A has the local Morera property with respect to the domain B, if for each function $f \in L_{loc}(B)$ the following relation

$$\int_{\partial(\lambda A)} f(z)dz = 0 \qquad (1)$$

for almost all $\lambda \in \mathrm{Mot}(\overline{A}, B)$

implies that $f \in Hol(B)$. We will denote by $Mor(B)$ the set of all domains with the Morera property with respect to B.

Morera type theorems have been studied by many authors (see [1] – [7]).

Of considerable interest is the case $B = B_r = \{z \in C : |z| < r\}$. One can ask the following question in this case.

Problem 1. For fixed A, describe the set $\{r > 0 : A \in Mor(B_r)\}$.

Morera's property intimately connected with the Pompeiu property (see Sections 5.4.1, 4.1.1, 4.1.2 in [6]). A large amount of research has gone into Pompeiu problem and related questions (see [6–11]).

It is easy to see that if $A \in Mor(B_r)$ for some $r > 0$ than $A \in Mor(B_R)$ for every $R > r$. The following problem arises in this connection.

Problem 2. Let $A \subset C$ be a compact set such that $A \in Mor(B_R)$ for some $R > r^*(A)$. Find

$$R(A) = \inf\{r > r^*(A) : A \in Mor(B_r)\}.$$

The problem 2 is solved for some family of set, which boundary consists of line segments and circular arcs.

In [7] the problem 2 has been considered for arbitrary circular sector. For $\alpha \in (0;\pi) \cup (\pi;2\pi)$, we set

$$S(\alpha) = \{z \in C : |z| \le 1, |\arg z| \le \alpha/2\},$$

$$P_1(\alpha) = \begin{cases} 5/8, \text{if } 0 < \alpha \le \arccos(4/5); \\ 1/(2\cos\alpha), \text{if } \arccos(4/5) < \alpha \le \pi/4; \\ \sin\alpha, \text{if } \pi/4 < \alpha \le \pi/2; \\ 1, \text{if } \pi/2 < \alpha < 2\pi. \end{cases}$$

In [7] the following theorem is obtained by the author of this paper.

Theorem A. *Let* $\alpha \in (0;\pi) \cup (\pi;2\pi)$ *be fixed and* $A = S(\alpha)$. *Then the following assertions hold:*

1) *If* $f \in L_{loc}(B_r)$, $r > P_1(\alpha)$ *and (1) is valid, then* $f \in Hol(B_r)$.
2) *If* $r^*(A) < r < P_1(\alpha)$ *then there exists a non-holomorphic function* $f \in C^\infty(B_r)$ *satisfying (1).*

From the results of [10] and [11] it follows that the next theorems B-C are valid.

Theorem B. *Let*

$$A = \{z \in C : |z| \le 1, |z - e^{-i\pi/6}| \le 1, |z - e^{i\pi/6}| \le 1.$$

Then the following assertions hold:

1) *If* $f \in L_{loc}(B_r)$, $r > 1$ *and (1) is valid, then* $f \in Hol(B_r)$.
2) *If* $1/\sqrt{3} < r < 1$ *then there exists a non-holomorphic function* $f \in C^\infty(B_r)$ *satisfying (1).*

For the next theorem, we introduce the following functions and number.

$$K_1(\alpha) = \frac{(2\sin\alpha + \cos\alpha - 1)^2 + \sin^2\alpha}{2(2\sin\alpha + \cos\alpha - 1)};$$

$$K_2(\alpha) = 0,5\sqrt{5 - 2\cos\alpha - 3\cos^2\alpha}.$$

Let us set $\alpha_0 \in (0;\pi)$ such that $K_1(\alpha_0) = K_2(\alpha_0)$. We also put

$$P_2(\alpha) = \begin{cases} K_1(\alpha), \text{if } 0 < \alpha < \alpha_0; \\ K_2(\alpha), \text{if } \alpha_0 \le \alpha \le \arccos(-1/7); \\ 1 - \cos\alpha, \text{if } \arccos(-1/7) < \alpha < \pi. \end{cases}$$

Theorem C. *Let* $\alpha \in (0;\pi)$ *be fixed and*

$A = \{z \in C : |z| \le 1, |\arg z| \le \alpha, \mathrm{Re}\, z \ge \cos\alpha\}$.

Then the following assertions hold:

(1) *If* $f \in L_{loc}(B_r)$, $r > P_2(\alpha)$ *and (1) is valid, then* $f \in Hol(B_r)$.
(2) *If* $\alpha \ge \alpha_0$ *and* $r^*(A) < r < P_2(\alpha)$ *then there exists a non-holomorphic function* $f \in C^\infty(B_r)$ *satisfying (1).*

As for one set the Morera property is in the interest for the family of sets. We will say that the set family $\{A_j\}_{j=1}^m$ has the Morera property in the domain B if for each function $f \in L_{loc}(B)$ the condition

$$\int_{\partial(\lambda_j A_j)} f(z)dz = 0$$

$$\text{for all } j = 1..m \text{ and almost all } \lambda_j \in Mot(\overline{A}_j, B) \qquad (2)$$

implies that $f \in Hol(B)$.

If the set family $\{A_j\}_{j=1}^m$ has the Morera property in the domain B we will denote it by $\{A_j\}_{j=1}^m \in Mor(B)$.

In this case the problem 2 is transforming to

Problem 3. Let $\{A_j\}_{j=1}^m$ be a family of compact set $A_j \subset C$. Find

$$R\left(\{A_j\}_{j=1}^m\right) = \inf\left\{r : \{A_j\}_{j=1}^m \in Mor(B_r)\right\}.$$

It is easy to see that for arbitrary set family $\{A_j\}_{j=1}^m$ the inequality

$$R\left(\{A_j\}_{j=1}^m\right) \le \min\left\{R(A_1),...R(A_m)\right\} \qquad (3)$$

is valid.

In this regard the question arises, for what family $\{A_j\}_{j=1}^m$ we can put in (3) the sign «<», and when the sign «=»?

In this paper the problem 3 is for the first time solved for the some families of circular sectors.

The main results of the paper are the following theorems.

Theorem 1. *Let* $\{\alpha_j\}_{j=1}^{m} \subset (0;\pi)\cup(\pi;2\pi)$, $\alpha_{\min} = \min\{\alpha_j, j=1..m\}$, $S_j = S(\alpha_j)$. *Then the following assertions hold:*

1) *If* $f \in L_{loc}(B_r)$, $r > P_1(\alpha_{\min})$ *and*

$$\int_{\partial(\lambda_j S_j)} f(z)dz = 0$$

for all $j = 1..m$ and almost all $\lambda_j \in Mot(S_j, B_r)$) (4)

then $f \in Hol(B_r)$.

2) *If* $r < P_1(\alpha_{\min})$ *then there exists a non-holomorphic function* $f \in C^\infty(B_r)$ *satisfying (4).*

Theorem 2. *Let* $\alpha_1 \in (0;\arccos(20\sqrt{1762}/881))$ *be an arbitrary,* $S_1 = 1{,}6S(\alpha_1)$, $S_2 = S(\pi/2)$. *Then the following assertions hold:*

1) *If* $f \in L_{loc}(B_r)$, $r > \sqrt{1762}/50$ *and (4) is valid for* $m = 2$, *then* $f \in Hol(B_r)$.

2) *If* $r < \sqrt{1762}/50$ *then there exists a non-holomorphic function* $f \in C^\infty(B_r)$ *satisfying (4) for* $m = 2$.

Let us note that theorem 1 gives us an example of the set family, for which inequality (3) becomes the equality, because the function $P_1(\alpha)$ is increasing on $\alpha \in (0;\pi)\cup(\pi;2\pi)$. On the other hand, we have $\sqrt{1762}/50 \approx 0{,}84 < 1$ and $\arccos(20\sqrt{1762}/881) \approx 17{,}65° < \arccos(4/5)$. So theorem 2 shows that there exists the family with Morera property in the circle such that no one of sets from the family has no Morera property in this circle.

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