UDC 519.6

MATHEMATICAL MODELING OF DISCONTINUOUS PROCESSES BY MEANS OF DISCONTINUOUS SPLINES IN A CASE WHEN SPLINE KNOTS DO NOT COINCIDE WITH RUPTURES

Lytvyn O.N., Pershina Y.I.

Ukrainian Engineering and Pedagogical Academy, Kharkov, Ukraine

Discontinuous approximational splines for approach of discontinuous functions are constructed and investigated. The algorithm of search of ruptures of one variable function by means of its approach constructed by the discontinuous approximational spline is developed. The algorithm of optimum definition of knots of the approaching discontinuous spline is developed and the corresponding examples are given.

KEY WORDS: discontinuous processes, approximational splines, knots.

МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ РАЗРЫВНЫХ ПРОЦЕССОВ С ПОМОЩЬЮ ДИСКРЕТНЫХ СПЛАЙНОВ В СЛУЧАЕ, КОГДА УЗЛЫ СПЛАЙНА НЕ СОВПАДАЮТ С РАЗРЫВАМИ

Литвин О.Н., Першина Ю.И.

В работе сконструированы и исследованы дискретные аппроксимирующие сплайны для дискретных функций. Разработан алгоритм поиска разрывов для функции одной переменной с помощью дискретных аппроксимирующих сплайнов. Предложен алгоритм оптимального определения узлов дискретных аппроксимирующих сплайнов и приведены соответствующие примеры.

КЛЮЧЕВЫЕ СЛОВА: дискретне процессы, аппроксимация сплайнами, узлы.

МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ РОЗРИВНИХ ПРОЦЕСІВ ЗА ДОПОМОГОЮ ДИСКРЕТНИХ СПЛАЙНІВ У ВИПАДКУ, КОЛИ ВУЗЛИ СПЛАЙНА НЕ ЗБІГАЮТЬСЯ З РОЗРИВАМИ

Литвін О.М., Пєршіна Ю.І.

В роботі сконструйовані та досліджені дискретні апроксимаційні сплайни для дискретних функцій. Розроблен алгоритм пошука розривів для функції однієї змінної за допомогою дискретних апроксимаційних сплайнів. Запропонован алгоритм оптимального визначення вузлів дискретних апроксимаційних сплайнів і наведені відповідні приклади.

КЛЮЧОВІ СЛОВА: дискретні процеси, апроксимація сплайнами, вузли.

1. Introduction. Research problems of discontinuous functions arise much more often, than research problems of continuous functions. For example, at research of internal structure of a body of the person it is useful to consider its heterogeneity, that is different density in different parts of a body (bones, heart, a stomach, a liver etc. have different density). I.e. the body density is function with ruptures of the first sort on system of surfaces which separate its different parts. Works [1–3] are devoted research of discontinuous functions, and approach of continuous functions is considered, for example, in works [4–6].

In [1] the problem of uniform approach of continuous and continuous-differentiated functions has been investigated by discontinuous one-variable splines. The approach of one variable continuous function by piece-constant functions when the continuous and differentiated functions come nearer splines of degree a zero is presented in [5,6]. As to approach of discontinuous functions the general methods of spline-approximation and spline-interpolation of discontinuous functions by means of discontinuous splines are unknown to authors.

Authors had already been investigated some questions of approach of discontinuous functions by discontinuous splines for one variable functions [7]–[8], and for two variables functions [9].

Those methods which have already been constructed suppose that ruptures of approached function are known, and consequently they coincide with ruptures of an approaching spline. In the given work the method of approaching of one-variable discontinuous function by discontinuous splines when ruptures of approached function still need to be discovered is developed. The algorithm of an optimum choice of knots of an approaching discontinuous spline is also offered.

© Lytvyn O.N., Pershina Y.I., 2014.

2. Problem formulation. Let function of one variable $f(x)$ on an interval $[a,b]$ with possible ruptures of the first sort is set, and it is not known, where they are. The work purpose is construction of an discontinuous approximating spline for approach of the discontinuous function constructed on set knots x_k, $k=\overline{1,n}$, dividing an interval $[a,b]$ on n parts, and working out of algorithm of an optimum choice of knots of an approaching spline.

3. Construction of discontinuous approximating spline. Let x_k, $k=\overline{1,n}$ – knots of an approaching approximating spline, and some of them coincide with point of discontinuities of the set discontinuous function. Let's be builds discontinuous approximating spline on each of segments in the form of the formula

$$\begin{gathered} S(x)=Sp_k(x,A)=C_k^{+}h_k(x)+C_{k+1}^{-}h_{k+1}(x), \\ [x_k,x_{k+1}],\ k=\overline{1,n-1}, \end{gathered} \tag{1}$$

where $h_k(x)$ is the polynomial basis with properties $h_i(x_{i'})=\delta_{i,i'}$, spline factors C_k^{+}, C_{k+1}^{-} are a method of least squares from a condition

$$\sum_{k=1}^{n-1}\int_{x_k}^{x_{k+1}}\left(f(t)-S(t,C)\right)^2 dt\to\min_C . \tag{2}$$

Theorem 1. The estimation of an error of approach of discontinuous function $f(x)$ by discontinuous linear approximating spline $S(x)$ of an aspect (1) on each interval of a partition looks like

– if $f(x)\in C^1[x_k,x_{k+1}]$, then

$$\begin{aligned} \|S(x)\|_{L_\infty[x_k,x_{k+1}]} &\le \max\{|f(x_k)|,|f(x_{k+1})|\}+ \\ &+\frac{x_{k+1}-x_k}{2}\cdot\|f'(x)\|_{L_\infty[x_k,x_{k+1}]}, \end{aligned} \tag{3}$$

– if $f(x)\in C^2[x_k,x_{k+1}]$, then

$$\begin{aligned} \|S(x)\|_{L_\infty[x_k,x_{k+1}]} &\le \max\{|f(x_k)|,|f(x_{k+1})|\}+ \\ &+\frac{(x_{k+1}-x_k)^2}{8}\cdot\|f''(x)\|_{L_\infty[x_k,x_{k+1}]}, \end{aligned} \tag{4}$$

$L_\infty[a,b]=\lim\limits_{p\to\infty}L_p[a,b]$.

Proof. According to the formula (1), discontinuous linear approximating spline on each interval of a partition becomes:

$$S(x)=C_k^{+}\frac{x-x_{k+1}}{x_k-x_{k+1}}+C_{k+1}^{-}\frac{x-x_k}{x_{k+1}-x_k},\ x\in[x_k,x_{k+1}].$$

Let's solve a minimisation problem:

$$J_k(C)=\int_{x_k}^{x_{k+1}}\left(f(x)-C_k^{+}\frac{x-x_{k+1}}{x_k-x_{k+1}}-C_{k+1}^{-}\frac{x-x_k}{x_{k+1}-x_k}\right)^2dx\to$$
$$\to\min_C,\qquad k=\overline{1,n-1}.$$

Let's write out system of the linear algebraic equations $\dfrac{\partial J_k(C)}{\partial C_k^{+}}=0,\ \dfrac{\partial J_k(C)}{\partial C_{k+1}^{-}}=0$ concerning unknown persons C_k^{+}, C_{k+1}^{-}:

$$\begin{cases} \displaystyle\int_{x_k}^{x_{k+1}}2\cdot\left(f(x)-C_k^{+}\frac{x-x_{k+1}}{x_k-x_{k+1}}-C_{k+1}^{-}\frac{x-x_k}{x_{k+1}-x_k}\right)\times \\ \displaystyle\qquad\times\left(-\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)dx=0, \\ \displaystyle\int_{x_k}^{x_{k+1}}2\cdot\left(f(x)-C_k^{+}\frac{x-x_{k+1}}{x_k-x_{k+1}}-C_{k+1}^{-}\frac{x-x_k}{x_{k+1}-x_k}\right)\times \\ \displaystyle\qquad\times\left(-\frac{x-x_k}{x_{k+1}-x_k}\right)dx=0, \end{cases}$$

$$\begin{cases} \displaystyle C_k^{+}\int_{x_k}^{x_{k+1}}\frac{(x-x_{k+1})^2}{(x_k-x_{k+1})^2}dx+C_{k+1}^{-}\int_{x_k}^{x_{k+1}}\frac{(x-x_k)(x-x_{k+1})}{(x_{k+1}-x_k)(x_k-x_{k+1})}dx= \\ \displaystyle\qquad=\int_{x_k}^{x_{k+1}}\frac{f(x)(x-x_{k+1})}{x_k-x_{k+1}}dx, \\ \displaystyle C_k^{+}\int_{x_k}^{x_{k+1}}\frac{(x-x_{k+1})(x-x_k)}{(x_k-x_{k+1})(x_{k+1}-x_k)}dx+C_{k+1}^{-}\int_{x_k}^{x_{k+1}}\frac{(x-x_k)^2}{(x_{k+1}-x_k)^2}dx= \\ \displaystyle\qquad=\int_{x_k}^{x_{k+1}}\frac{f(x)(x-x_k)}{x_{k+1}-x_k}dx. \end{cases}$$

In the received system we will make replacement $C_{k+1}^{-}=f(x_{k+1}-0)+\varepsilon_{k+1}$, $C_k^{+}=f(x_k+0)+\varepsilon_k$, and we will substitute an interpolational polynomial of Lagrange with an error term $R(x)$. As a result we will receive following expressions for integral members of system

$$\int_{x_k}^{x_{k+1}}f(x)\cdot\frac{x-x_{k+1}}{x_k-x_{k+1}}dx=\frac{1}{3}(x_{k+1}-x_k)f(x_k+0)+$$
$$+\frac{1}{6}(x_{k+1}-x_k)f(x_{k+1}-0)+\int_{x_k}^{x_{k+1}}R(x)\cdot\frac{x-x_{k+1}}{x_k-x_{k+1}}dx,$$

$$\int_{x_k}^{x_{k+1}}\frac{x-x_{k+1}}{x_k-x_{k+1}}\frac{x-x_{k+1}}{x_k-x_{k+1}}dx=\frac{1}{(x_k-x_{k+1})^2}\times$$
$$\times\int_{x_k}^{x_{k+1}}(x-x_{k+1})^2dx=\frac{1}{(x_k-x_{k+1})^2}\left.\frac{(x-x_{k+1})^3}{3}\right|_{x_k}^{x_{k+1}}=$$
$$=\frac{x_{k+1}-x_k}{3},$$

$$\int_{x_k}^{x_{k+1}} \frac{x-x_k}{x_{k+1}-x_k}\frac{x-x_{k+1}}{x_k-x_{k+1}}dx =$$

$$= -\frac{1}{(x_{k+1}-x_k)^2}\left(\frac{x^3}{3}-(x_{k+1}+x_k)\frac{x^2}{2}+x_k x_{k+1}x\right)\Bigg|_{x_k}^{x_{k+1}} =$$

$$= -\frac{1}{(x_{k+1}-x_k)}\left(\frac{x_{k+1}^2+x_k x_{k+1}+x_k^2}{3}-\right.$$

$$\left.-(x_{k+1}+x_k)\frac{x_{k+1}+x_k}{2}+x_k x_{k+1}\right) = -\frac{2x_{k+1}^2+2x_k x_{k+1}}{6(x_{k+1}-x_k)}-$$

$$-\frac{+2x_k^2-3x_{k+1}^2-6x_k x_{k+1}-3x_k^2+6x_k x_{k+1}}{6(x_{k+1}-x_k)} =$$

$$= -\frac{1}{x_{k+1}-x_k}\left(\frac{2x_k x_{k+1}-x_k^2-x_{k+1}^2}{6}\right) =$$

$$= \frac{(x_{k+1}-x_k)^2}{6(x_{k+1}-x_k)} = \frac{1}{6}(x_{k+1}-x_k),$$

$$\int_{x_k}^{x_{k+1}} \frac{x-x_k}{x_{k+1}-x_k}\frac{x-x_k}{x_{k+1}-x_k}dx = \frac{1}{(x_{k+1}-x_k)^2}\int_{x_k}^{x_{k+1}}(x-x_k)^2 dx =$$

$$= \frac{1}{(x_{k+1}-x_k)^2}\frac{(x-x_k)^3}{3}\Bigg|_{x_k}^{x_{k+1}} = \frac{1}{3}(x_{k+1}-x_k);$$

$$\int_{x_k}^{x_{k+1}} f(x)\cdot\frac{x-x_{k+1}}{x_k-x_{k+1}}dx = \frac{1}{3}(x_{k+1}-x_k)f(x_k+0)+$$

$$+\frac{1}{6}(x_{k+1}-x_k)f(x_{k+1}-0)+\int_{x_k}^{x_{k+1}} R(x)\cdot\frac{x-x_{k+1}}{x_k-x_{k+1}}dx;$$

$$\int_{x_k}^{x_{k+1}} f(x)\cdot\frac{x-x_k}{x_{k+1}-x_k}dx = \frac{1}{6}(x_{k+1}-x_k)f(x_k+0)+$$

$$+\frac{1}{3}(x_{k+1}-x_k)f(x_{k+1}-0)+\int_{x_k}^{x_{k+1}} R(x)\cdot\frac{x-x_k}{x_{k+1}-x_k}dx.$$

Let's receive system of a following aspect:

$$\begin{cases} \frac{1}{3}(x_{k+1}-x_k)\cdot\varepsilon_k+\frac{1}{6}(x_{k+1}-x_k)\cdot\varepsilon_{k+1} = \\ \qquad = \int_{x_k}^{x_{k+1}} R(x)\frac{x-x_{k+1}}{x_k-x_{k+1}}dx, \\ \frac{1}{6}(x_{k+1}-x_k)\cdot\varepsilon_k+\frac{1}{3}(x_{k+1}-x_k)\cdot\varepsilon_{k+1} = \\ \qquad = \int_{x_k}^{x_{k+1}} R(x)\frac{x-x_k}{x_{k+1}-x_k}dx. \end{cases} \tag{5}$$

For the analysis of right members of the received system we will use formulas from paper [7]

1) when $f(x)\in C^1[x_k,x_{k+1}]$ then

$$\|f(x)-S(x)\|_{L_\infty[x_k,x_{k+1}]} \le \frac{x_{k+1}-x_k}{2}\cdot\|f'(x)\|_{L_\infty[x_k,x_{k+1}]};$$

2) when $f(x)\in C^2[x_k,x_{k+1}]$ then

$$\|f(x)-S(x)\|_{L_\infty[x_k,x_{k+1}]} \le \frac{(x_{k+1}-x_k)^2}{8}\cdot\|f''(x)\|_{L_\infty[x_k,x_{k+1}]}.$$

Using the labels $\|\varepsilon\| = \max\{\varepsilon_k^+,\varepsilon_{k+1}^-\}$, we will rewrite system (5) in an aspect

1. if $f(x)\in C^1[x_k,x_{k+1}]$, then

$$\begin{cases} \frac{1}{3}(x_{k+1}-x_k)\cdot\|\varepsilon\| \le \frac{1}{6}(x_k-x_{k+1})\cdot\|\varepsilon\|+ \\ \qquad +\|f'(x)\|_{L_\infty[x_k,x_{k+1}]}\left(\frac{x_{k+1}-x_k}{2}\right)^2, \\ \frac{1}{6}(x_{k+1}-x_k)\cdot\|\varepsilon\| \le \frac{1}{3}(x_k-x_{k+1})\cdot\|\varepsilon\|+ \\ \qquad +\|f'(x)\|_{L_\infty[x_k,x_{k+1}]}\left(\frac{x_{k+1}-x_k}{2}\right)^2. \end{cases} \Rightarrow$$

$$\Rightarrow \|\varepsilon\| \le \frac{x_{k+1}-x_k}{2}\cdot\|f'(x)\|_{L_\infty[x_k,x_{k+1}]};$$

2. if $f(x)\in C^2[x_k,x_{k+1}]$, then

$$\begin{cases} \frac{1}{3}(x_{k+1}-x_k)\cdot\|\varepsilon\| \le \frac{1}{6}(x_k-x_{k+1})\cdot\|\varepsilon\|+ \\ \qquad +\|f''(x)\|_{L_\infty[x_k,x_{k+1}]}\frac{(x_{k+1}-x_k)^3}{16}, \\ \frac{1}{6}(x_{k+1}-x_k)\cdot\|\varepsilon\| \le \frac{1}{3}(x_k-x_{k+1})\cdot\|\varepsilon\|+ \\ \qquad +\|f''(x)\|_{L_\infty[x_k,x_{k+1}]}\frac{(x_{k+1}-x_k)^3}{8}. \end{cases} \Rightarrow$$

$$\Rightarrow \|\varepsilon\| \le \frac{(x_{k+1}-x_k)^2}{8}\cdot\|f''(x)\|_{L_\infty[x_k,x_{k+1}]}.$$

From the received inequalities also there is a theorem proof.

The theorem 1 is proved.

Remark 1. If function $f(x)=a(const)$ comes nearer an discontinuous linear spline of an aspect (1) be method of least squares, then estimation (3) is exact, if function looks like $f(x)=ax+b$, then in an estimation (4) as the equality is reached.

Remark 2. If approached function $f(x)$ is piecewise-linear or piecewise-constant function with point of discontinuities $x=x_k$, $k=\overline{1,n}$ and is approached by its piecewise-linear spline $S(x)$ defined by formulas (1)–(2), we will receive precisely approached function, that is $S(x)=f(x)$.

Remark 3. If $C_k^+=C_k^-=S(x_k)$, $k=\overline{1,n-1}$, then constructed discontinuous approximating spline is a continuous linear approximating spline.

4. Algorithm of an optimum choice of knots of an approaching discontinuous spline. Let x_k, $k=\overline{1,n}$ – knots of an approaching spline which do not coincide with given function ruptures. Let's state algorithm of determination of ruptures of a given function.

Pitch 1. We build discontinuous approximating spline S(x) on the given knots x_k, $k=\overline{1,n}$ to the formula (1), which on each interval of a partition can have a different analytical appearance $S_k(x,C)$ with unknown persons C_k^+, C_{k+1}^-, $k=\overline{1,n-1}$.

Pitch 2. We discover a matrix of unknown factors C of a spline from a condition (2).
At carrying out of computing experiment for minimisation standard procedure of system of computer mathematics MathCad – under a title was used

$$\text{Minimize}(\sum_{k=1}^{n}\int_{x_k}^{x_{k+1}}\left(f(t)-S(t,C)\right)^2 dt, C).$$

After substitution of the discovered factors in a spline (1) we will receive discontinuous a spline which consists of functions $S_k(x)=Sp_k(x,C)$, $k=\overline{1,n-1}$.

Pitch 3. On each of intervals $[x_k, x_{k+1}]$, $k=\overline{1,n-1}$ we calculate values

$$J_k^* = \max_{x_k\le x\le x_{k+1}} J_k(x), \quad J_k(x)=\left|f(x)-S_k(x)\right|.$$

Definition. If $\left|\lim_{x\to x_q^+} f(x) - \lim_{x\to x_q^-} f(x)\right| < \varepsilon$, then function $f(x)$ we will name ε - continuous in a point x_q.

Pitch 4. If conditions are satisfied

1. $J_q<\varepsilon$, $J_{q+1}<\varepsilon$, when ε – the set accuracy of approach;
2. $f(x)$ is ε - continuous in a point x_{q+1},

then knot x_{q+1} is deleted from reviewing.

Pitch 5. From all J_k^* it is chosen maximum value and we divide an interval to which posesses this maximum value, for example $W\in[x_r, x_{r+1}]$, $r<n$, in halves, that is the new knot $x^* = x_r + \frac{x_{r+1}-x_r}{2}$ is introduced into set of knots of a spline.

Pitch 6. On new set of knots again we build an approximating spline under the formula (1) and under the formula (2) it is discovered an unknown matrix C of factors. And further we check condition performance

$$\max_{x\in[a,b]}\left|f(x)-Sp(x)\right|<\varepsilon,$$

where ε is the set accuracy of the approach. If this condition is fulfilled, we have received an optimum choice of knots of an approached spline among which there are also given function ruptures. If the specified condition is not fulfilled, we come back to the pitch 3

Example 1. Let in area $D=[0,1]$ function is set (fig.1)

$$f(x)=\begin{cases}4x^2, & x\in(0,0.5];\\ 2, & x\in(0.5,1].\end{cases}.$$

There is a function has a rupture of the first sort in the point $x=0.5$.

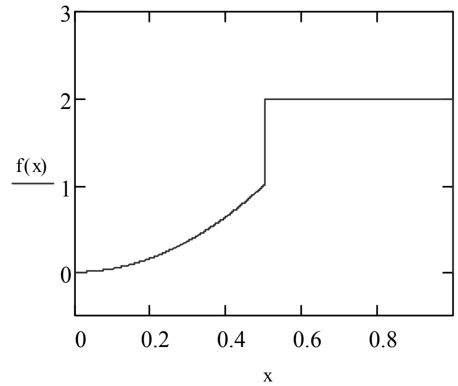


Fig.1. Graphic presentation of function $f(x)$

We choose spline knots so that they did not coincide with given function ruptures $x_1=0$, $x_2=0.3$, $x_3=0.6$, $x_4=1$. Let's construct discontinuous an approximating linear spline in the form of the formula (1) which in our case will look like

$$S_k(x,C)=C_{k,1}\frac{x-x_{k+1}}{x_k-x_{k+1}}+C_{k,2}\frac{x-x_k}{x_{k+1}-x_k}, \quad x_k\le x\le x_{k+1},\ k=\overline{1,3}, \tag{6}$$

where matrix elements C it is discovered from a condition (2). That is we build discontinuous spline with point of discontinuities in spline knots (fig. 2). Let's set accuracy of approach $\varepsilon=0{,}01$. The results received by 2–24 iterations are presented in fig.3. After 24 iterations the spline S(x) has approached the set function with accuracy ε. Thus have optimum chosen knots of a spline which equal

$x_1=0$, $x_2=0.075$, $x_3=0.15$, $x_4=0.3$, $x_5=0.5$, $x_6=1$

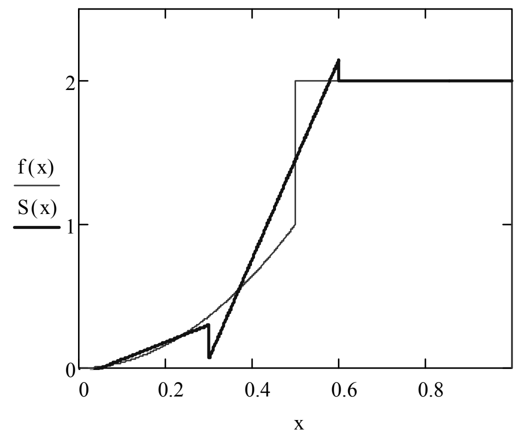
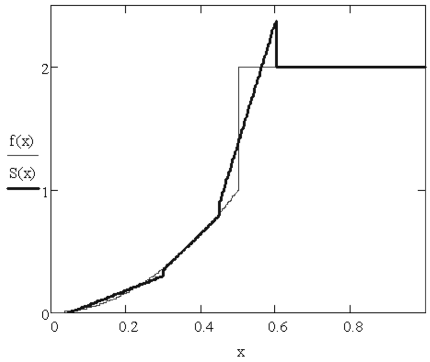
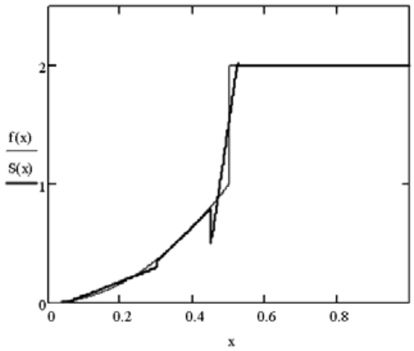


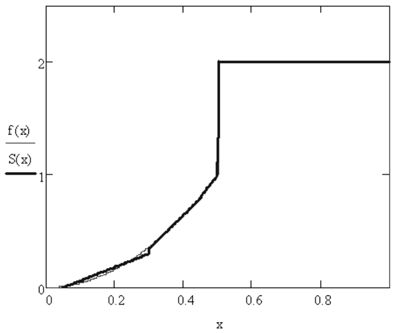
Fig.2. Graphic aspect of approached function (low-fat line) and the constructed spline (a fat line)



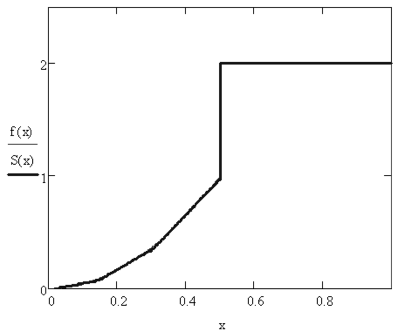
a



b



c



d

Fig.3. Graphic aspect of approached function (low-fat line) and the constructed spline (a fat line) on: a) 2 iterations; b) 3 iterations; c) 18 iterations; d) 24 iterations.

Example 2. Let us define in the area $D = [0,1]$ the function set as (fig.4)

$$g(x) = \begin{cases} -12x^2 + 2, & x \in [0, 0.4] \\ 3 - x, & x \in (0.4, 0.7] \\ 1, & x \in (0.7, 0.1] \end{cases}.$$

The function has a rupture of the first sort in a point $x = 0.4$, $x = 0.7$.

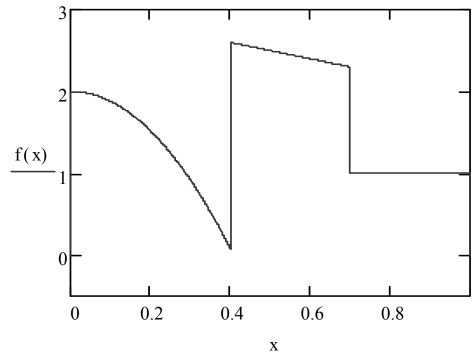
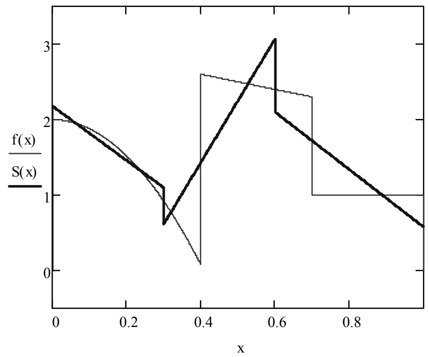
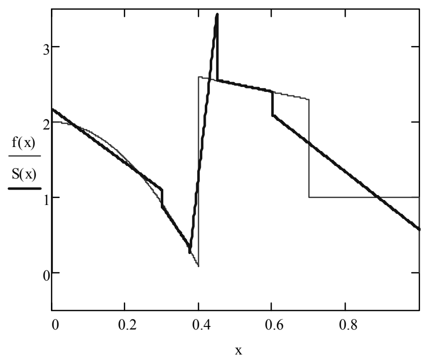


Fig.4. Graphic representation of function $g(x)$.

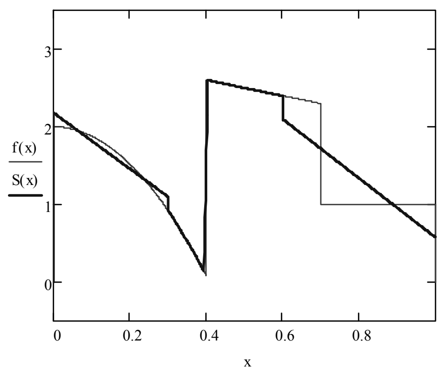
The spline knots have been chosen as $x_1 = 0$, $x_2 = 0.3$, $x_3 = 0.6$, $x_4 = 1$. Let's construct discontinuous approximating linear spline in the form of the formula (6). Let's set accuracy the same as in the example 1. Some intermediate results of the approach are shown in fig.5.



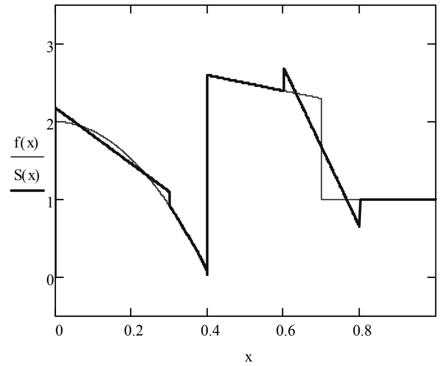
a



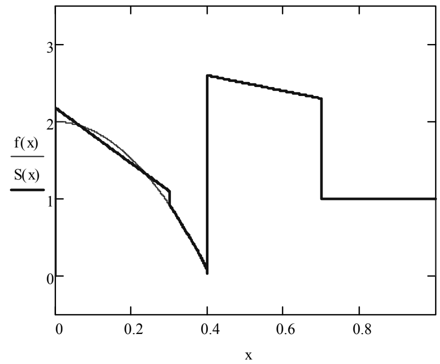
b



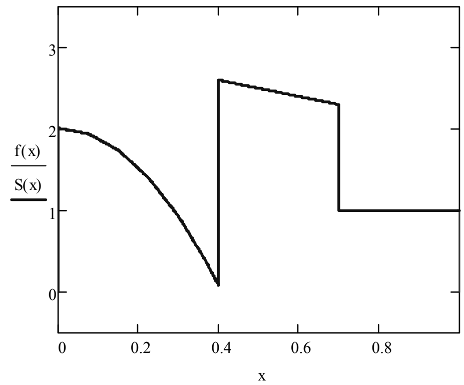
c



d



e



f

Fig.5. Graphic aspect of approached function (low-fat line) and the constructed spline (a fat line) on: a) 1 iterations; b) 5 iterations; c) 10 iterations; d) 15 iterations; e) 30 iterations; f) 37 iterations.

That is on 37 iterations the spline S(x) has approached the set function g(x) with accuracy ε. Thus we have optimum chosen knots of an approaching spline among which there are also ruptures of the set function

$$x_1 = 0,\ x_2 = 0.075,\ x_3 = 0.15, x_4 = 0.225,$$

$$x_5 = 0.3,\ x_6 = 0.375,\ x_7 = 0.4,\ \ x_8 = 0.7, x_9 = 1.$$

5. Conclusions. In this work the algorithm of search of ruptures of one variable function is developed. The algorithm is developed on the basis of approach of discontinuous function by discontinuous approximating linear spline. Also the algorithm of optimum definition of knots of an approaching discontinuous spline is developed. Examples which confirm the stated theory are resulted.

Further investigations the optimum peep of knots of discontinuous spline for the functions of two variables with rectangularing range of definition are planned.

Acknowledgments. The work is supported by the Ukrainian foundation DFFD.

REFERENCES

1. Popov B. A. *Uniform approach by splines*. K.: Nauk. Dumka. – 1989. – 272p. (in Russian).
2. Petrovskaya N.B. Approximation of explosive decisions for one class of schemes of a high order *Mathem. modelling*. – 2005.– v. 17, N1.– P.79–92 (in Russian).
3. Arnold D.N. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Analysis*.– 2002.– v.39, N5.– P. 1749–1779.
4. Lytvyn O.N. *Interlineation functions and Some appendices*. Osnova, Kharkov. – 2002. – 544p. (in Ukrainian).
5. Stechkin S.B., Subbotin U.N. *Splines in calculus mathematics.* Nauka, Mowcow. – 1976 (in Russian).
6. Zavialov U.S., Kvasov B.I., Miroshnichenko V.L. *Methods of splines-functions*. Nauka, Moscow. – 1976 (in Ukrainian).
7. Lytvyn O.N., Pershina Y.I. Mathematical modelling of processes which have ruptures, by means of discontinuous interpolational splines *Artificial intelligence.* – 2011. – N2. – P.152 – 158 (in Ukrainian).
8. Lytvyn O.N., Pershina Y.I. Approach of discontinuoua function of two variables by means of discontinuous splines of two variables (rectangular elements). *Computer mathematics*. *Kiev.* – 2011. – N1. – P.96 – 105 (in Russian).
9. Lytvyn O.N., Pershina Y.I. Approach of discontinuous functions of two variables with ruptures of the first sort on lines of a triangulation of two-dimensional area. *Operating systems and cars. Kiev.* – 2011. – N5. – P.34–47. (in Russian).