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MAPPING OF THE GAUSS TYPE IN GENERALIZATION OF THE BORSUK PROBLEM TO SOME BANACH SPACES

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The analogue of the well-known problem of K.Borsuk on the decomposition of a bounded subset of R^n_W in n+1 parts of smaller diameter is considered. The sufficient condition for the possibility of such decomposition is obtained. The presented results significantly extend the known class for which the conjecture of K.Borsuk is valid.

KEY WORDS: Borsuk problem, Banach spaces, Gauss type map.

ОТОБРАЖЕНИЕ ГАУССА В ОБОБЩЕНИИ ГИПОТЕЗЫ БОРСУКА НА НЕКОТОРЫЕ БАНАХОВЫ ПРОСТРАНСТВА

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В работе рассматривается аналогия известной гипотезы К.Борсука о разложении ограниченного подмножества R^n_W на n+1 часть с меньшими диаметрами. Получено достаточное условие возможности такой декомпозиции. Представленные результаты существенно расширяют известный класс, для которого гипотеза Борсука верна.

КЛЮЧЕВЫЕ СЛОВА: гипотеза Борсука, Банаховы пространства, отображение Гаусса.

ВІДОБРАЖЕННЯ ГАУСА В УЗАГАЛЬНЕННІ ГІПОТЕЗИ БОРСУКА НА ДЕЯКІ БАНАХОВІ ПРОСТОРИ

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В роботі розглядається аналогія відомої гіпотези Борсука о розкладенні обмеженої підмножини R^n_W на n+1 частину з мешьшими діаметрами. Отримано достатнью умову можливості такої декомпозиції. Отримані результати істотно розширюють відомий клас, для якого гіпотеза Борсука вірна.

КЛЮЧОВІ СЛОВА: гіпотеза Борсука, Банахові простори, відображення Гауса.

1.Introduction. One of the central problems in combinatorial geometry is a problem of decomposition of figures into parts of smaller diameter. This problem is known as Borsuk's problem. In 1933, K. Borsuk advanced the following hypothesis.

Borsuk's hypothesis 1. *For a given set* $G \subset R^n$ *of diameter* d*, there exist the subsets* $K_0, K_1, ..., K_n$ *each of which has a diameter smaller than* d*, and they form a covering* G.

Hypothesis proved in R^2 by K.Borsuk at 1933 [1,2] and in R^3 by H.Eggleston in 1955[3], later by B.Grunbaum [4], A. Heppes [5] in 1957. To obtain their results K.Borsuk, B.Grunbaum and A. Heppes used the only geometrical methods.

However, in spaces with dimension more than three there are no results without conditions on set. Because of the fact that the given approach turned out unsuitable for the derivation of a final positive or negative answer. At 1991 J. Kahn and G. Kalai constructed the first counterexample to the hypothesis [6]. They showed that, for all dimensions more than 2014, there exists a figure for which Borsuk's assumption is erroneous. Then a series of works concerning the construction of counterexamples in the spaces R^n with lower dimensions. The last result in this direction was obtained by A. Hinrichs and C. Richter for $n \geq 298$ [7].

In this connection, the description of sets for which the given partition is valid becomes of primary importance. There are many results in this direction of development of the problematic. The first one of this sequence was the theorem of H. Hadwiger which was obtained in 1946 [8].

Theorem 1.1. *Any convex body from* R^n *that has a smooth boundary can be divided into* $n+1$ *parts of smaller diameter.*

It is seen from this assertion that namely the irregular points on the boundary of a set create the greatest difficulty for the partition into parts with smaller diameters. To obtain this result Hadwiger used a spherical Gauss map. He put into correspondence the sphere S^{n-1} with the smooth boundary of a set $G \subset R^n$. Then, he divided S^{n-1} to the $n+1$ parts of smaller diameter and thanks to the antipodal property of the spherical Gauss map received a required decomposition of G. Note that this method was used Eggleston to confirm the hypothesis in R^3 unconditionally.

The main results for sets with nonsmooth boundary are obtained by R. Anderson and Klee [9] in 1952, and then V. Boltyanskii [10] in 1960. This was followed by a series of partial results. In 2011 we significantly extended results of V.Boltyanskii by using mapping of the Gauss type with the antipodal property [11].

Also investigated extensions of the Borsuk's problem to the spaces with non-Euclidean metric. For the subsets of a two-dimensional normalized space whose unit ball is a parallelogram, this problem was solved by B. Grunbaum [12] and by V. Boltyanskii and V. Soltan [13] for any two-dimensional normalized space. In author's work [14] strong sufficient conditions in R^n space with norm $|x|_p^p = \sum_{i=1}^{n} |x_i|^p$ for $1 < p < \infty$ are obtained.

In this article we try to generalize methods obtained in author's works [11,14] to the n-dimensional banach spaces. We will construct a norm-forming set which make applying our methods.

2.Main result. Let us construct a vector space R_W^n with the norm generated by the set W. For this we define the Minkowski functional as follows:

$$F_W(x) = \inf\{r > 0 \big| \tfrac{x}{r} \in W\}, x \in R^n,$$

and if for some value of $x \in R^n$ such infimum does not exist, then F_W is considered equal to infinity. Let also the Minkowski F_W functional takes finite positive values for all $x \in R^n$ unequal to zero. Then F_W specifies the norm $|\bullet|_W$ in a linear vector space, moreover W is the unit ball in this space[15].

Let us introduce a function $\varsigma : \partial W \to \partial G$, where ∂W — the unit sphere in R_W^n, G is a set of constant width. For $\theta \in \partial W$, assume

$$\varsigma(\theta) = x, \text{ where } x, y \in \partial G, |x - y|_W = diamG$$

and

$$x - y = \theta diamG \quad (1)$$

Thus x, y are points at which attained the diameter of G. This function is an analogue of the spherical Gauss map. But if the Gauss map is the only meaningful for sets with smooth boundary, then the function is defined for any set of constant width.

Also, we need a function to count the number of diameters that pass through a given point. Let $\chi : G \to Z_+ \cup \{\infty\}$ where Z_+ is the set of non-negative integers, and G is a set of constant width. For all $x \in G$, assume $\chi(x)$ to be equal the number of diameters that pass through x. Let us agree $\chi(x) = \infty$, if from x pass infinity diameters.

Below the main result of the work is presented.

Theorem 2.1. *Let* $\mathrm{W} \subset \mathrm{R}^n$ *be a closed strictly convex centrally symmetric set, and suppose that the Minkowski* $\mathrm{F_W}$ *functional takes finite positive values for all* $\mathrm{x} \in \mathrm{R}^n$ *unequal to zero. Let* $\mathrm{R_W^n}$ *be the space with the norm formed by the functional* $\mathrm{F_W}$.

Let also $\mathrm{G} \subset \mathrm{R_W^n}$ *be a set of constant width,* $\mathrm{EP(G)} = \{\mathrm{x} | \mathrm{x} \in \partial\mathrm{G} : \chi(\mathrm{x}) = \infty\}$, L *a set of all connected components of* $\Theta = \{\theta | \varsigma(\theta) \in \mathrm{EP}\}$. *In addition, there are the following conditions:*

1) for all $\mathrm{U} \in \mathrm{L}$,

$$\sup_{\alpha,\beta \in \mathrm{U}} |\alpha - \beta|_\mathrm{W} < \inf_{\mathrm{x,y} \in \partial\mathrm{W},(\mathrm{x,y})=0} |\mathrm{x} - \mathrm{y}|_\mathrm{W} - \sigma$$

for some $\sigma > 0$ *independent of U;*

2) if $\mathrm{U,V} \in \mathrm{L}$, $\mathrm{U} \setminus \tilde{\mathrm{V}} \neq \varnothing$ *and* $\tilde{\mathrm{V}} \setminus \mathrm{U} \neq \varnothing$, *where* $\tilde{\mathrm{V}} = \{-\beta | \beta \in \mathrm{V}\}$, *then* $\mathrm{int}(\tilde{\mathrm{V}}) \cap \mathrm{U} = \varnothing$.

Then G *can be divided into* $\mathrm{n}+1$ *parts of smaller diameter.*

To prove Theorem 2.1, we construct a system of $n+1$ subsets covering ∂W each of which have diameter less then $diamW$. Next we apply the function ς defined in (1). Thus we construct a system of $n+1$ subsets covering ∂G each of which has the diameter less then $diamG$. Of course not every system of subsets of W has this property. But the methods of their construction which are represented in work [14] allow us to obtain needed system of subsets of ∂G.

Let us agree to call the set which can be divided into n+1 parts of smaller diameter as Borsuk's set. The theorem 2.1 shows that the class of Borsuk's sets in R_W^n spaces no less than in the classical space R^n with Euclidean metric. So, as a consequence of this theorem we can obtain the results by Hadwiger and by Boltyansky on R_W^n spaces.

Corollary 2.1. *Let* $\mathrm{W} \subset \mathrm{R}^n$ *be a closed strictly convex centrally symmetric set. Any set* $\mathrm{G} \subset \mathrm{R_W^n}$ *of constant width that has a smooth boundary can be divided into* $\mathrm{n}+1$ *parts of smaller diameter.*

Corollary 2.2. *Let* $\mathrm{W} \subset \mathrm{R}^n$ *be a closed strictly convex centrally symmetric set. Any set* $\mathrm{G} \subset \mathrm{R_W^n}$ *of constant width that has no more than* n *irregular points in the boundary can be divided into* $\mathrm{n}+1$ *parts of smaller diameter.*

To prove corollary 2.1 it suffices to note that $\mathrm{EP(G)} = \varnothing$ for constant width set's with a smooth boundary. This means that conditions 1 and 2 of Theorem 2.1 are performed automatically.

To prove corollary 2.2 it important to note that all the points of the set $EP(G)$ must be isolated. This means that for all $U \in L$, one has

$$\sup_{\alpha,\beta \in U} |\alpha - \beta|_W < A(W),$$

where $A(W) < \inf_{x,y \in \partial W,(x,y)=0} |x-y|_W$. For example, for $\partial W = S^{n-1}$, in the classical Euclidean space R^n, $A(W) = 1$ and $\inf_{x,y \in S^{n-1},(x,y)=0} |x-y|_W = \sqrt{2}$.

Nevertheless, due to the restrictions imposed on the set G in Theorem2.1, it seems an artificial result of this statement. This is partly true. Generally, for an arbitrary Banach spaces the imbedding $G' \subseteq G$ is incorrect for some $G' \subset R^n_W$, where $G \subset R^n_W, \mathrm{diam}G' = \mathrm{diam}G$ and G is a set of constant width. This question is considered in details in [16] by H. Eggleston. However, we are not seen quite an arbitrary Banach space, but the space in which the normforming set is closed, strictly convex, and centrally symmetric. For spaces of this type the imbedding $G' \subseteq G$ which we have discussed above becomes true, if W satisfies an additional condition presented in [17] by R. Karasev.

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