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ON THE THEORY OF VARIATIONS FOR THE BELTRAMI EQUATION

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A brief overview of the resent results in the theory of variations for classes of regular solutions to the degenerate Beltrami equation with constraints of the set theoretic and integral types for the complex coefficients is given. On this basis, the variational principles of maximum and other necessary conditions of extremum are formulated and applications to one of the main equations of the mathematical physics are given.

KEY WORDS: variational principles, degenerate Beltrami equation, conditions of extremum.

О ВАРИАЦИОННОЙ ТЕОРИИ ДЛЯ УРАВНЕНИЯ БЕЛЬТРАМИ

Гутлянский В.Я., Ломако Т.В., Рязанов В.И.

В работе приведен краткий обзор последних результатов вариационной теории для классов регулярных решений вырожденного уравнения Бельтрами с органичениями теории множеств и интегрального типов для случая комплексных коэффициентов. На этой основе сформулированы вариационные принципы максимума и другие необходимые условия экстремума. Обсуждаются приложения к одному из основных уравнений математической физики.

КЛЮЧЕВЫЕ СЛОВА: вариационные принципы, вырожденное уравнение Бельтрами, условия экстремума.

О ВАРІАЦІЙНІЙ ТЕОРІЇ ДЛЯ РІВНЯННЯ БЕЛЬТРАМІ

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В роботі наведено короткий огляд останніх результатів варіаційної теорії для класів регулярних рішень виродженого рівняння Бельтрамі з обмеженями теорії множин і інтегрального типів для випадку комплексних коефіцієнтів. На цій основі сформульовані варіаційні принципи максимуму і інші необхідні умови екстремуму. Обговорюються застосування до одного з основних рівнянь математичної фізики.

КЛЮЧОВІ СЛОВА: варіаційні принципи, вироджене рівняння Бельтрамі, умови екстремуму.

1. Introduction. The existence problem for the degenerate Beltrami equation

$$f_{\bar z} = \mu(z)\cdot f_z \qquad (1)$$

i.e., when the condition of the uniform ellipticity $|\mu(z)| \le k < 1$ is replaced by a weaker condition $|\mu(z)| < 1$ a.e. and $\|\mu(z)\|_\infty = 1$, is currently an active area of research, see e.g. the monographs [1–3] and the surveys [4, 5]. We say that an orientation preserving homeomorphism $f : D \to \mathbf{C}$, satisfying (1.1) a.e. in D, is a *regular solution* to the Beltrami equation if $f \in W^{1,1}_{loc}$ and the Jacobian $J_f(z) > 0$ a.e. in D, see [6]. In this case we also say that μ is the *complex dilatation* of the mapping f. New theorems on the existence of the regular solutions for (1), when the degeneracy of the ellipticity is controlled by the dilatation coefficient

$$K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \in L^1_{loc},$$

see e.g. the monograph [1] and the survey [4], as well as the corresponding theorems on compactness and convergence, developed in [1, 7–9], allowed us to originate a general, constructive variational procedure for investigations of extermal problems in the relevant classes of mappings. This paper contains a brief overview of our results from [10] devoted to the theory of variations for classes of regular solutions to the degenerate Beltrami equation with constraints of the set-theoretic and integral types for the complex coefficient $\mu(z)$. Our approach to the construction of variations is based on the convexity of the set of complex coefficients, which turns out to be the common property for the compact classes of regular solutions to the Beltrami equation, see, e.g., [11–14].

Recall that the variational theory for quasiconformal mappings, that corresponds to the case when $|\mu(z)| \le k < 1$ was substantially developed by P.P. Belinskii, S.L. Krushkal, R. Kühnau, H. Renelt, M. Schiffer and G. Schober and others, see, e.g., [15–19] and the references therein.

2. Variational procedure. For the illustration of general variational procedure, we consider first a general class H of regular solutions $f:\overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of the Beltrami equation (1.1), normaized by $f(0)=0, f(1)=1, f(\infty)=\infty$, when complex dilatations μ vary over a *convex set* $\boldsymbol{M}$ of measurable functions satisfying $K_\mu \in L^1_{loc}$.

Theorem 2.1. *Let* $f \in H$ *and have complex dilatation* $\mu \in \boldsymbol{M}$. *Then for every* $\omega \in \boldsymbol{M}$ *the mappings*

$$f_\varepsilon(\zeta) = f(\zeta) - \frac{\varepsilon}{\pi}\int_{\mathbf{C}} (\omega(z)-\mu(z))\,\phi(f(z),f(\zeta))\times \left(f_z/\overline{f_z}\right)J_f(z)\,dxdy + o(\varepsilon,\zeta) \in H, \tag{2}$$

for each $\varepsilon \in [0,1/2)$. *Here*

$$\phi(w,w') = \frac{1}{w-w'}\cdot\frac{w'}{w}\cdot\frac{w'-1}{w-1}, \tag{3}$$

and $o(\varepsilon,\zeta)/\varepsilon \to 0$ *locally uniformly with respect to* $\zeta \in \mathbf{C}$.

The proof of Theorem 2.1 is based on the theory of composition operators, see e.g. [20] and [21], and the known theorem on the differentiability of families of quasiconformal mappings with respect to a parameter [22, Chapter 5].

Remark 2.1. The normalization $f(z) = z + o(1)$, where $o(1) \to 0$ as $z \to \infty$, implies the simplest variational derivative for mappings whose characteristic is equal to zero in a neighborhood of the infinity:

$$\phi(w,w') = \frac{1}{(w-w')}.$$

3. Necessary conditions for extremum. Let $\Omega : H \to \mathbf{R}$ be a functional defined over the class H. We also require that Ω is Gâteaux differentiable, i.e.

$$\Omega(f^*) = \Omega(f) + t\,\mathrm{Re}\iint_{\mathbf{C}} g\,d\kappa + o(t)$$

for every admissible variation $f^* = f + tg + o(t)$ in the class H. Here κ is a finite complex Radon measure with compact support. Furthermore, we suppose that the kernel $\phi(w,f(z))$ is locally integrable with respect to the product measure $dxdy \otimes d\kappa$ and

$$A(w) = \frac{1}{\pi}\iint_{\mathbf{C}} \phi(w,f(z))\,d\kappa(z) \ne 0$$

almost everywhere.

Theorem 2.1 allows us to prove a maximum principle which states, that under the above conditions, the complex dilatation μ of an extreme mapping f belongs to the set of the extreme points of the convex set $\boldsymbol{M}$. For the illustration of this principle we specify the set $\boldsymbol{M}$ and start with the case when μ are subordinated to some set-theoretic constrains.

3.1. Recall that the family of compact sets $M(z) \subseteq \mathbf{D}$, $z \in \mathbf{C}$, where $\mathbf{D}$ stands for the unit disk centered at the origin, is called *measurable in the parameter* z, if, for any closed set $M_0 \subseteq \mathbf{C}$, the set $E_0 = \{z \in \mathbf{C} : M(z) \subseteq M_0\}$ is measurable by Lebesgue. Denote by $\boldsymbol{M}_M$ the class of all measurable functions

$$\mu(z) \in M(z) \quad \text{a.e.}$$

and, correspondingly, by H^*_M, the class of all regular homeomorphisms $f:\overline{\mathbf{C}} \to \overline{\mathbf{C}}$ with complex dilatations $\mu \in \boldsymbol{M}_M$ and the normalizations $f(0)=0$, $f(1)=1, f(\infty)=\infty$.

Theorem 3.1. *Let* $\max\Omega$ *over the class* H^*_M *be attained on a mapping* $f \in H^*_M$ *with complex dilatation* $\mu(z)$. *Then* $\mu(z) \in \partial M(z)$ *for almost all* $z \in \mathbf{C}$.

By $\omega_\mu(z)$, we denote a *cone of admissible directions* for the set M(z) at the point $\mu(z)$, i.e., the set of all $\omega \in \mathbf{C}, \omega \ne 0$, such that $\mu(z) + \varepsilon\omega \in M(z)$ for all $\varepsilon \in [0,\varepsilon_0]$ and some $\varepsilon_0 > 0$.

Theorem 3.2. *Let* $\max\Omega$ *over the class* H^*_M *be attained on a mapping* $f \in H^*_M$ *with complex dilatation* $\mu(z)$. *Then* $\mathrm{Re}\,\omega A(f(z))f_z^2 \ge 0$ *for almost all* $z \in \mathbf{C}$ *and all* ω *in the cone of admissible directions* $\omega_\mu(z)$.

Corollary 3.1. *If at every point of* $\partial M(z)$ *for almost all* $z \in \mathbf{C}$ *there exists the tangent, then* $n(z)A(f(z))f_z^2 \ge 0$ *a.e., where* n(z) *is the unit vector of the internal normal to* $\partial M(z)$ *at the point* $\mu(z)$.

Considering the important partial case, when

$$M(z) = \{\kappa \in \mathbf{C} : |\kappa - c(z)| \le k(z)\} \subseteq \mathbf{D},$$
$$Q(z) := \frac{1+k(z)+|c(z)|}{1-k(z)-|c(z)|}, \tag{4}$$

and c(z), k(z) are measurable functions, we arrive at the following statement.

Corollary 3.2. *Let* $M(z), z \in \mathbf{C}$, *be the family of circles (3.1) and* $Q(z) \in L^1_{loc}$, *and let* $\max\Omega$ *over the class* H^*_M *be attained on a mapping* $f \in H^*_M$ *with complex dilatation* $\mu(z)$. *Tthen* f *satisfies the equation*

$$f_{\bar z} = c(z)f_z - k(z)\frac{\overline{A(f(z))}}{|A(f(z))|}\overline{f_z}.$$

If $Q(z) \in L^{\infty}$, we recognize the well-known necessary conditions of the extremum for quasiconformal mappings.

3.2. We proceed with the necessary conditions for the extremum over some classes of mappings with constraints of the integral type on the dilatation coefficient $K_{\mu}(z)$. Let us remark that similar classes of quasiconformal mappings in the mean were studied by L. Ahlfors, P.A. Biluta, B.V. Bojarski, A. Gol'berg, V.I. Kruglikov, S.L. Krushkal, V.S. Kud'yavin, R. Kühnau, M. Perovich, I.N. Pesin, and others (see, e.g., references in [11, Chapter 13] and [3, Chapter 12]).

Let $\Phi : \overline{\mathbf{R}^{+}} \to \overline{\mathbf{R}^{+}}$ be a nondecreasing convex function. By $\boldsymbol{F}^{\Phi}$, we denote the class of all regular solutions $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of the Beltrami equation normalized by $f(0)=0, f(1)=1, f(\infty)=\infty$, with complex dilatation μ such that

$$\int_{\mathbf{C}} \Phi\left(K_{\mu}(z)\right) dS(z) \le 1 .$$

Here $dS(z)$ stands for the *element of the spherical area* in $\overline{\mathbf{C}}$. By $\boldsymbol{M}^{\Phi}$, we denote the corresponding class of complex characteristics. It is easily to verify that $\boldsymbol{M}^{\Phi}$ is a convex set.

Theorem 3.3. *Let* $\Phi : \overline{\mathbf{R}^{+}} \to \overline{\mathbf{R}^{+}}$ *be a nondecreasing convex function with* $\Phi(Q) \neq 0$*, where* $Q = \sup_{\Phi(t)<\infty} t$. *Then, for any mapping* $f \in \boldsymbol{F}^{\Phi}$ *such that* $\max_{\boldsymbol{F}^{\Phi}} \Omega = \Omega(f)$*, the dilatation* $K_{\mu}(z)$ *satisfies the equality*

$$\int_{\mathbf{C}} \Phi(K_{\mu}(z))\, dS(z) = 1$$

and

$$f_{\bar{z}} = -k(z)\, \frac{\overline{A(f(z))}}{|A(f(z))|}\, \overline{f_z} ,$$

where $k(z) = (K_{\mu}(z)-1)/(K_{\mu}(z)+1)$.

4. An example of application. Let D be a domain in $\overline{\mathbf{C}}$. We study the second order degenerate elliptic equation in divergence form

$$\operatorname{div}(K(z)\operatorname{grad} U) = 0 , \qquad (5)$$

which is important in applied mathematics. The degeneracy of the ellipticity is controlled by the condition

$$\int_{D} \Phi\left(K(z)\right) dS(z) < \infty \qquad (6)$$

for the strictly convex function $\Phi : [1,\infty] \to [0,\infty]$ satisfying the condition

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \qquad (7)$$

for $\delta > \Phi(0)$.

Recall that the function $\Phi : \overline{\mathbf{R}^{+}} \to \overline{\mathbf{R}^{+}}$ is said to be *strictly convex* if Φ is convex, non-decreasing and $\lim_{t\to\infty} \Phi(t)/t = \infty$.

By a weak solution to (4.1) we understand a function U that possesses the *stream function* V such that the complex valued function $F = U + iV, z = x + iy$, has the first generalized derivatives and satisfies the Beltrami equation of the second kind

$$F_{\bar{z}} = -\frac{K(z)-1}{K(z)+1}\, \overline{F_z} ,$$

a.e. in D.

An appropriate selection of the functional Ω, see [10], together with Theorem 3.3 allows us to give the following existence, representation and regularity theorem for the weak solution to the equation (4.1).

Theorem 4.1. *Let* D *be a domain in* $\overline{\mathbf{C}}$, *and let* $K : D \to [1,\infty]$ *be a measurable function satisfying (4.2) for the strictly convex function* $\Phi : [1,\infty] \to [0,\infty]$ *with condition (4.3). Then, for any* $z_0 \in D \setminus \{\infty\}$, *there exists a weak solution of (4.1) that is representable in the form* $U(z,z_0) = \ln|f(z) - f(z_0)|^{-1}$, *where* f *is a regular solution of the equation*

$$f_{\bar{z}} = -\frac{K(z)-1}{K(z)+1} \cdot \frac{\overline{f(z)-f(z_0)}}{f(z)-f(z_0)}\, \overline{f_z}$$

in $\overline{\mathbf{C}}$ *normalized by* $f(0)=0$, $f(1)=1$, $f(\infty)=\infty$.

If, in addition, the function $K(z)$ *is approximatively continuous at the point* z_0 *and*

$$\limsup_{r\to 0} \frac{1}{|D(z_0,r)|} \int_{D(z_0,r)} \Phi\left(K(z)\right) dm(z) < \infty,$$

where $D(z_0,r) = \{z \in \mathbf{C} : |z - z_0| < r\}$, *then*

$$\lim_{z\to z_0} \frac{U(z,z_0)}{\ln|z-z_0|^{-1}} = \frac{1}{K(z_0)} .$$

For the uniformly elliptic case see, e.g., [23–25] and [11, Chapter 14].

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