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MULTIPLE-SCALE METHOD AND GAS-DYNAMIC-TYPE EQUATIONS

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The purpose of the paper is to obtain regular gas-dynamic-type equations suitable for description the space relaxation relative to uniform state. The treatment is based on the space linearized Boltzmann equation and the multiple-scale method, and it gives the same result when applied to the Navier-Stokes equations. The equations for leading terms beyond the Navier-Stokes determine the vortex dissipation and dispersion of the sound waves in gas.

KEY WORDS: gas dynamics, Boltzmann equations, Navier-Stokes equations.

МЕТОД МНОГИХ МАСШТАБОВ И УРАВНЕНИЯ ГАЗОДИНАМИЧЕСКОГО ТИПА

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Целью данной работы является получение регулярных уравнений газодинамического типа, подходящих для описания пространственной релаксации относительно однородного состояния. Исследование основано на пространственно линеаризованном уравнении Больцмана и методе многих масштабов, который дает тот же результат будучи примененным к уравнениям Навье-Стокса. Уравнения для старших членов относительно описываемых уравнениями Навье-Стокса определяют диссипацию вихрей и дисперсию звуковых волн в газе.

КЛЮЧЕВЫЕ СЛОВА: газовая динамика, уравнения Больцмана, уравнения Навье-Стокса.

МЕТОД БАГАТЬОХ МАСШТАБІВ ТА РІВНЯННЯ ГАЗОДИНАМІЧНОГО ТИПУ

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Метою даної роботи є отримання регулярних рівнянь газодинамічного типу, які придатні для опису просторової релаксації відносно однорідного стану. Дослідження засноване на просторово лінеаризованому рівнянні Больцмана і методі багатьох масштабів, який дає той же результат при застосуванні до рівнянь Нав'є-Стокса. Рівняння для старших членів щодо описуваних рівняннями Нав'є-Стокса визначають дисипацію вихорів і дисперсію звукових хвиль в газі.

КЛЮЧОВІ СЛОВА: газова динаміка, рівняння Больцмана, рівняння Нав'є-Стокса.

1. Introduction. The task of the present paper is to obtain the regular gas-dynamic-type equations suitable for description the space relaxation relative to uniform state and define the limits of its application. The treatment is based on the space linearized Boltzmann equation and the multiple-scale method.

The Navier-Stokes equations are basic equations of hydrodynamics. However, this continuum approach has certain restrictions to describe gas behavior. First of all, gas is a system of a very large number of moving and interacting molecules. The continuum mechanics methods in this case can be used only in the continuum limit of small Knudsen number $\varepsilon = l_S / L_S$, where l_S is the mean free path of the molecules, L_S is a typical dimension of the problem. Furthermore, the viscosity and heat conduction terms are included in the hydrodynamical equations on the basis of the experimental Hook-Newton and Fourier laws [1]. As it is known, the transport coefficients in gas are turn out to be proportional to the free path [2]. As a result, the nondimensional Navier-Stokes equations depend linearly upon the small Knudsen number and become invalid to obtain a correct solution up to order ε^2 .

Moreover, the solutions of the Navier-Stokes equations in the form of power series in ε exhibit unbounded growth with time and cease to describe correctly the dissipative processes [3,4]. On the other hand, the kinetic theory is based on the molecular structure of the gas and valid at any Knudsen number particular in the continuum limit**.** For this reason, one of the tasks of a theory based on the Boltzmann equation is to develop some approximate macroscopic model and define the limits of its applications. D. Hilbert has obtained the principal results here. [5], but the straightforward expansions in powers of the small Knudsen number break down for long times [4,6]. It is believed that some of the troubles can be avoided by the Chapman-Enskog expansion and that this method yields the Navier-Stokes-Fourier equations [5]. However, the disadvantages of this expansion are well known [7–12]. Y. Sone investigates the asymptotic analysis of the boundary-value problems in detail in the monograph [13]. The most important contribution of Chapman and Enskog is the additional expansion of the time derivative. However, the idea to expand equations while leaving the density, velocity and temperature unexpanded is not able to provide the uniform expansion. The multiple-scale methods are good adapted to study a wide variety of the physical processes that are occurring at different rates [4.6].

2. The problem and asymptotic method. Let us consider the time evolution of small disturbances based on the space linearized dimensionless Boltzmann equation

$$\varepsilon\left(\frac{\partial\phi}{\partial t}+\mathbf{c}\cdot\frac{\partial\phi}{\partial\mathbf{r}}\right)=\mathrm{L}\phi,$$
$$\mathrm{L}\phi=\int f_0(\mathbf{c}_1)\left[\phi(\mathbf{c}_1')+\phi(\mathbf{c}')-\phi(\mathbf{c}_1)-\phi(\mathbf{c})\right]\mathrm{gbdbd}\beta\mathrm{d}\mathbf{c}_1, \quad (1)$$
$$f_0=(2\pi)^{-3/2}\exp\left(-\frac{c^2}{2}\right), \quad f=f_0\phi.$$

The hydrodynamic values are defined as

$$n=\int f_0\phi\mathrm{d}\mathbf{c},\ \mathbf{v}=\int \mathbf{c}f_0\phi\mathrm{d}\mathbf{c},$$
$$3p=\int c^2 f_0\phi\mathrm{d}\mathbf{c},\ \ p=n+T. \quad (2)$$

Let us complete the Hilbert-like expansion with multivariable technique. Then we assume

$$\varphi=\varphi_0+\varepsilon\varphi_1+\varepsilon^2\varphi_2+... \quad (3)$$

Corresponding to the (3), the macroscopic variables (2) are also expanded

$$\rho=\rho_0+\varepsilon\rho_1+\varepsilon^2\rho_2+...\,, \quad (4)$$

where ρ represents n, v, p and ρ_k represents $n_k, \mathbf{v}_k, p_k$

It should be noted, that asymptotic expansion is uniformly valid, i.e. the solution of the problem may be represented by the first terms of the asymptotic expansion, if throughout the range of interest

$$\varepsilon\rho_k \angle\angle \rho_{k-1} \quad (5)$$

However, this condition doesn't mean that each ρ_k must be small. It means that each term in (4) must be small as compared with the preceding term.

It is the rule rather than exemption that the conditions (5) are not fulfilled and, in particular, break down for long times. Therefore, the essential point of any correct asymptotic procedure is the detection at each step of the sources of secular behavior and elimination of secular terms.

As remarked above, we rely upon the multivariable technique [4,6] and introduce the sequence of new time variables $t_k=\varepsilon^k t$ that

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}+\varepsilon^2\frac{\partial}{\partial t_2}+... \quad (6)$$

3. Leading term of the expansion. Substituting (3), (6) into (1) gives a sequence of the linear integral equations $\mathrm{L}\varphi_k=Q_k$

$$Q_1=\frac{\partial\varphi_0}{\partial t_0}+\mathbf{c}\cdot\nabla\varphi_0 \qquad Q_2=\frac{\partial\varphi_0}{\partial t_1}+\frac{\partial\varphi_1}{\partial t_0}+\mathbf{c}\cdot\nabla\varphi_1$$
$$Q_3=\frac{\partial\varphi_0}{\partial t_2}+\frac{\partial\varphi_1}{\partial t_1}+\frac{\partial\varphi_2}{\partial t_0}+\mathbf{c}\cdot\nabla\varphi_2 \qquad Q_0=0 \quad (7)$$

We can find φ_k, provided that Q_k orthogonal to the five collision invariants

$$\int\psi_r f_0 Q_k \mathrm{d}\mathbf{c}=0 \qquad \psi_r=1,\mathbf{c},c^2 \quad (8)$$

The general solutions of (7) are

$$\varphi_k=\alpha_k+\beta_{ki}c_i+\gamma_k g_k+h_k \quad (9)$$

with partial solutions h_k. Arbitrary functions $\alpha_k, \beta_{ki}, \gamma_k$, can be expressed with the help (2)–(4) in terms of $n_k, p_k, \mathbf{v}_k$.

Since the lowest order $\mathrm{L}\varphi_0$, we have

$$\varphi_0=p_0+\mathbf{v}_0\cdot\mathbf{c}+\left(\frac{c^2}{2}-\frac{5}{2}\right)T_0 \quad (10)$$

The solvability conditions (8) for $\mathrm{L}\varphi_1=Q_1$ lead to the linearized Euler equations with respect to t_0

$$\frac{\partial n_0}{\partial t_0}+\mathrm{div}\mathbf{v}_0=0,$$
$$\frac{\partial\mathbf{v}_0}{\partial t_0}+\nabla p_0=0, \quad (11)$$
$$\frac{\partial p_0}{\partial t_0}+\frac{5}{3}\mathrm{div}\mathbf{v}_0=0.$$

The set (11) gives the equations for $\mathbf{v}_0$, p_0 and the adiabatic relation

$$\frac{\partial^2\mathbf{v}_0}{\partial t_0^2}-\frac{5}{3}\nabla\mathrm{div}\Delta\mathbf{v}_0=0\,,\quad \frac{\partial^2 p_0}{\partial t_0^2}-\frac{5}{3}\Delta p_0=0 \quad (12)$$

$$\frac{\partial s_0}{\partial t_0}=0 \qquad s_k=p_k-\frac{5}{3}n_k \quad (13)$$

Since ρ_0 are functions of another time variables t_k, it is necessary to consider the next approximation.
Taking in account (11), we obtain from (7)

$$\mathrm{L}\varphi_1=\left(\frac{c^2}{2}-\frac{5}{2}\right)\mathbf{c}\cdot\nabla T_0+\overset{\circ}{\mathbf{cc}}:\frac{\partial\mathbf{v}_0}{\partial\mathbf{r}} \quad (14)$$

The solution of the integral equation (14) is

$$\varphi_1=p_1+\mathbf{v}_1\cdot\mathbf{c}+\left(\frac{c^2}{2}-\frac{5}{2}\right)T_1+h_1 \quad (15)$$

For the Maxwell's molecules we have [2]

$$h_1=-\frac{2}{5}\lambda\left(\frac{c^2}{2}-\frac{5}{2}\right)\mathbf{c}\cdot\nabla T_0-\mu\overset{\circ}{\mathbf{cc}}:\frac{\partial\mathbf{v}_0}{\partial\mathbf{r}} \qquad \lambda=\frac{15}{4}\mu\,,$$

where heat conduction and viscosity coefficients λ and μ are given by the solutions of appropriate integral equations [2]

For the next approximation the solvability conditions are

$$\frac{\partial n_1}{\partial t_0} + \operatorname{div}\mathbf{v}_1 = -\frac{\partial n_0}{\partial t_1}$$

$$\frac{\partial \mathbf{v}_1}{\partial t_0} + \nabla p_1 = -\left[\frac{\partial \mathbf{v}_0}{\partial t_1} - \mu\left(\Delta\mathbf{v}_0 + \frac{1}{3}\nabla \operatorname{div}\mathbf{v}_0\right)\right] \tag{16}$$

$$\frac{\partial p_1}{\partial t_0} + \frac{5}{3}\operatorname{div}\mathbf{v}_1 = -\left(\frac{\partial p_0}{\partial t_1} - \frac{2}{3}\lambda\Delta T_0\right)$$

Since s_0 doesn't depend on t_0, we have

$$\frac{\partial}{\partial t_0}\left(s_1 + \mu\nabla\cdot\mathbf{v}_0\right) = 0 \quad \frac{\partial s_0}{\partial t_1} - \frac{3}{2}\mu\Delta s_0 = 0 \tag{17}$$

Then the set (16) leads to inhomogeneous equations

$$\frac{\partial^2 \mathbf{v}_1^*}{\partial t_0^2} - \frac{5}{3}\nabla \operatorname{div}\mathbf{v}_1^* = -2\frac{\partial}{\partial t_0}\left[\frac{\partial \mathbf{v}_0}{\partial t_1} - \mu\left(\Delta\mathbf{v}_0 + \frac{1}{6}\nabla\operatorname{div}\mathbf{v}_0\right)\right],$$

$$\frac{\partial^2 p_1}{\partial t_0^2} - \frac{5}{3}\Delta p_1 = -2\frac{\partial}{\partial t_0}\left(\frac{\partial p_0}{\partial t_1} - \frac{7}{6}\mu\Delta p_0\right), \tag{18}$$

$$\mathbf{v}_1^* = \mathbf{v}_1 - \frac{9}{10}\mu\nabla s_0.$$

The terms on the right-hand side produce solutions proportional to t_0. In order that $\varepsilon\rho_1$ in (4) are to be small corrections to ρ_0 up to times of $O\left(\varepsilon^{-1}\right)$, we require

$$\frac{\partial \mathbf{v}_0}{\partial t_1} = \mu\left(\Delta\mathbf{v}_0 + \frac{1}{6}\nabla\operatorname{div}\mathbf{v}_0\right), \quad \frac{\partial p_0}{\partial t_1} = \frac{7}{6}\mu\Delta p_0 \tag{19}$$

As a consequence, the solvability conditions (16) are transformed into

$$\frac{\partial}{\partial t_0}\left(s_1 + \mu\operatorname{div}\mathbf{v}_0\right) = 0 \qquad \frac{\partial \mathbf{v}_1}{\partial t_0} + \nabla p_1 = \frac{1}{6}\mu\nabla\operatorname{div}\mathbf{v}_0$$

$$\frac{\partial p_1}{\partial t_0} + \frac{5}{3}\operatorname{div}\mathbf{v}_1 = \frac{3}{2}\mu\Delta s_0 - \frac{1}{6}\mu\Delta p_0 \tag{20}$$

4. Dissipative gas-dynamic-type equations for the leading term.

Now we can arrange the solvability conditions (11), (13), (17) and (19) in a set of gas-dynamical equations that determine the leading term for the times as large as $O\left(\varepsilon^{-1}\right)$

$$\frac{\partial s_0}{\partial t} = \frac{2}{5}\varepsilon\lambda\Delta s_0$$

$$\frac{\partial \mathbf{v}_0}{\partial t} = -\nabla p_0 + \varepsilon\mu\left(\Delta\mathbf{v}_0 + \frac{1}{6}\nabla\operatorname{div}\mathbf{v}_0\right) \tag{21}$$

$$\frac{\partial p_0}{\partial t} = -\frac{5}{3}\operatorname{div}\mathbf{v}_0 + \frac{7}{6}\varepsilon\mu\Delta p_0\,; \quad \varepsilon t \le 1$$

The equations (21) differ from the linearized Navier-Stokes system

$$\frac{\partial n}{\partial t} = -\operatorname{div}\mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \varepsilon\mu\left(\Delta\mathbf{v} + \frac{1}{3}\nabla\operatorname{div}\mathbf{v}\right),$$

$$\frac{\partial p}{\partial t} = -\frac{5}{3}\operatorname{div}\mathbf{v} + \frac{2}{3}\varepsilon\lambda\Delta T$$

In the classical Hilbert procedure the Navier-Stokes set equations never appear. However, the Navier-Stokes equations in the gas case contain itself the small parameter and can lead to singular solutions. As it can easily be proved, the two-time scale method applied to the singular Navier-Stokes set in the limiting case of small ε gives the asymptotic result for ρ_0 in full agreement with the regular set (21).

5. Gas-dynamic-equations up to order $\mathbf{Kn}^2$.

If it is necessary to obtain solutions up to order ε^2, neither the Navier-Stokes equation itself nor the set (21) are not suitable. We must proceed with analysis in a similar way as above and consider briefly the third approximation. After calculations we get

$$\frac{\partial n_2}{\partial t_0} + \operatorname{div}\mathbf{v}_2 = -\frac{\partial n_1}{\partial t_1} - \frac{\partial n_0}{\partial t_2}$$

$$\frac{\partial \mathbf{v}_2}{\partial t_0} + \nabla p_2 = -\left[\begin{matrix}\dfrac{\partial \mathbf{v}_1}{\partial t_1} - \mu\left(\Delta\mathbf{v}_1 + \dfrac{1}{3}\nabla\operatorname{div}\mathbf{v}_1\right) + \\ + \dfrac{\partial \mathbf{v}_0}{\partial t_2} - \dfrac{8}{15}\mu^2\Delta\nabla p_0\end{matrix}\right] - \frac{6}{5}\lambda\Delta\nabla s_0$$

Taking into account (11), (17), (19), we obtain

$$\frac{\partial p_2}{\partial t_0} + \frac{5}{3}\operatorname{div}\mathbf{v}_2 = -\left(\begin{matrix}\dfrac{\partial p_1}{\partial t_1} - \mu\Delta p_1 + \dfrac{\partial p_0}{\partial t_2} - \\ -\dfrac{7}{6}\mu^2\Delta\operatorname{div}\mathbf{v}_0\end{matrix}\right) + \frac{2}{5}\lambda\Delta s_1 \tag{22}$$

$$\frac{\partial}{\partial t_0}\left(s_2 + \mu\operatorname{div}\mathbf{v}_1 + \frac{3}{5}\mu^2\Delta p_0\right) = -\frac{\partial s_0}{\partial t_2} - \left(\frac{\partial}{\partial t_1} - \frac{3}{2}\mu\Delta\right)\left(s_1 + \mu\operatorname{div}\mathbf{v}_0\right) \tag{23}$$

$$\frac{\partial^2 \mathbf{v}_2^*}{\partial t_0^2} - \frac{5}{3}\nabla\operatorname{div}\mathbf{v}_2^* = -2\frac{\partial}{\partial t_0}\Big[\frac{\partial \mathbf{v}_1}{\partial t_1} - \mu(\Delta\mathbf{v}_1 + \frac{1}{6}\nabla\operatorname{div}\mathbf{v}_1) + \frac{\partial \mathbf{v}_0}{\partial t_2} - \frac{19}{120}\mu^2\Delta\nabla p_0\Big] \tag{24}$$

$$\mathbf{v}_2^* = \mathbf{v}_2 - \frac{9}{10}\mu\nabla s_1$$

The terms on the right-hand side of (23) are independent on the value of t_0 and make $\varepsilon^2 s_2$ the same order as εs_1 when $t_0 \to \varepsilon^{-1}$, unless

$$\frac{\partial s_0}{\partial t_2} + \left(\frac{\partial}{\partial t_1} - \frac{3}{2}\mu\Delta\right)\left(s_1 + \mu\operatorname{div}\mathbf{v}_0\right) = 0$$

However, as follows from (13), εs_1 becomes $O(s_0)$ when $t_0 \to \varepsilon^{-2}$, unless

$$\frac{\partial s_0}{\partial t_2} = 0 \qquad \left(\frac{\partial}{\partial t_1} - \frac{3}{2}\mu\Delta\right)\left(s_1 + \mu\operatorname{div}\mathbf{v}_0\right) = 0 \tag{25}$$

The terms on the right-hand side of (24) produce solutions proportional to t_0 and make $\varepsilon^2\mathbf{v}_2$ the same order as $\varepsilon\mathbf{v}_1$ when $t_0 \to \varepsilon^{-1}$, unless

$$\frac{\partial \mathbf{v}_1}{\partial t_1} - \mu\left(\Delta\mathbf{v}_1 + \frac{1}{6}\nabla\operatorname{div}\mathbf{v}_1\right) = -\left(\frac{\partial \mathbf{v}_0}{\partial t_2} - \frac{19}{120}\mu^2\Delta\nabla p_0\right)$$

Since the right-hand side of this relation gives for $\mathbf{v}_1^*$ the solution

$$t_1\left(\frac{\partial \mathbf{v}_0}{\partial t_2}-\frac{19}{120}\mu^2\Delta\nabla p_0\right),$$

the expansion $\mathbf{v}_0+\varepsilon\mathbf{v}_1$ breaks down for $t_0 \to \varepsilon^{-2}$. Therefore, we must put

$$\begin{aligned}&\frac{\partial \mathbf{v}_1}{\partial t_1}-\mu\left(\Delta\mathbf{v}_1+\frac{1}{6}\nabla\mathrm{div}\mathbf{v}_1\right)=0,\\&\frac{\partial \mathrm{v}_0}{\partial t_2}-\frac{19}{120}\mu^2\Delta\nabla p_0=0.\end{aligned}\tag{26}$$

Similarly we get

$$\begin{aligned}&\frac{\partial p_1}{\partial t_1}-\mu\frac{7}{6}\Delta p_1=0,\\&\frac{\partial p_0}{\partial t_2}-\frac{19}{72}\mu^2\Delta\mathrm{div}\mathbf{v}_0=0.\end{aligned}\tag{27}$$

We can arrange the solvability conditions (11), (13), (17), (19) and (26), (27) in a set of gas-dynamical equations that determine the leading term for times as large as $O\left(\varepsilon^{-2}\right)$. Replacing $\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}+\varepsilon^2\frac{\partial}{\partial t_2}$ by $\frac{\partial}{\partial t}$, we get

$$\begin{aligned}&\frac{\partial s_0}{\partial t}=\frac{2}{5}\varepsilon\lambda\Delta s_0,\\&\frac{\partial \mathbf{v}_0}{\partial t}=-\nabla p_0+\varepsilon\mu\left(\Delta\mathbf{v}_0+\frac{1}{6}\nabla\mathrm{div}\mathbf{v}_0\right)+\\&\qquad+\frac{19}{120}\varepsilon^2\mu^2\Delta\nabla p_0,\\&\frac{\partial p_0}{\partial t}=-\frac{5}{3}\mathrm{div}\mathbf{v}_0+\frac{7}{6}\varepsilon\mu\Delta p_0+\frac{19}{72}\varepsilon^2\mu^2\Delta\mathrm{div}\mathbf{v}_0,\end{aligned}\tag{28}$$

where $t \le \varepsilon^{-2}$.

Eqs (28) describe the time evolution of the different disturbances. For example, the ***div*** and ***rot*** operations allow us to split the problem and obtain, in particular,

$$\left(\frac{\partial}{\partial t}-\varepsilon\mu\Delta\right)\mathrm{rot}\mathbf{v}_0=0\tag{29}$$

which defines the vortex dissipation. From the other-hand, the potential part determines the damping and the dispersion of the sound wave in a gas.

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