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ON RECENT ADVANCES IN DIRICHLET PROBLEM FOR DEGENERATE BELTRAMI EQUATIONS

[1]**Bojarski B.V.**, [2]**Gutlyanski V.Ya.**, [2]**Ryazanov V.I.**

[1]*Institute of Mathematics of Polish Academy of Sciences, Warsaw, Poland*
2Institute of Applied Mathematics and Mechanics, NASU, Donetsk,Ukraine

The paper contains a brief overview of recent criteria on the existence of regular solutions for the Dirichlet problem to general Beltrami equations in arbitrary Jordan domains of the complex plane.

KEY WORDS: Dirichlet problem, Beltrami equations.

О ПОСЛЕДНИХ ДОСТИЖЕНИЯХ В РЕШЕНИИ ЗАДАЧИ ДИРИХЛЕ ДЛЯ ВЫРОЖДЕННОГО УРАВНЕНИЯ БЕЛЬТРАМИ

Боярский Б.В., Гутлянский В.Я., Рязанов В.И.

В работе приведен краткий обзор критериев существвоания регулярных решений задачи Дирихле для общих уравнений Бельтрами в производной Жордановой области на комплексной плоскости.

КЛЮЧЕВЫЕ СЛОВА: Задча Дирихле, уравнения Бельтрами.

ПРО ОСТАННІ ДОСЯГНЕННЯ У РОЗВЯЗАННІ ЗАДАЧІ ДІРІХЛЕ ДЛЯ ВИРОДЖЕНОГО РІВНЯННЯ БЕЛЬТРАМІ

Боярський Б.В., Гутлянскій В.Я., Рязанов В.І.

В роботі наведено стислий огляд критеріїв існування регулярних рішень задачі Дирихле для загальних рівнянь Бельтрамі в Жордановій області на комплексній площині.

КЛЮЧОВІ СЛОВА: задчие Дирихле, рівняння Бельтрамі.

1. Introduction.

The classical Dirichlet problem in a Jordan domain *D* for the uniformly elliptic Beltrami equation

$$\begin{cases} f_{\overline{z}} = \mu(z)\cdot f_z + \nu(z)\cdot \overline{f}_z, \quad z \in D, \\ \lim\limits_{z\to\varsigma} \operatorname{Re} f(z) = \varphi(\varsigma), \quad \forall\, \varsigma \in \partial D, \end{cases} \tag{1}$$

i.e., when the measurable coefficients satisfy the inequality $|\mu(z)|+|\nu(z)| \le k < 1$ and $\varphi(\zeta)$ stands for a continuous function, have been studied by the first author in 1957 in [1], see also [2]. A function *f* is said to be a *regular solution* of the Dirichlet problem (1) in a domain *D,* if *f* realizes a continuous, discrete and open mapping $f : D \to C$ of the Sobolev class $W^{1,1}_{loc}$ and such that the Jacobian $J_f(z) \ne 0$ a.e. in *D.*

It was shown in [1], see Proposition 1 below, that the regular solution *f,* normalized in an appropriate way, can be found by the method of successive approximations. First, one assumes that *D* is the unit disk $B = \{z : |z| < 1\}$ and $\varphi(z) = \operatorname{Re}\Phi(z)$ for $|z| = 1$ where $\Phi(z)$, $\operatorname{Im}\Phi(1) = 0$, is an analytic function in B satisfying the assumption $\Phi'(z) \in L^p$ for some $p > 2$. Then the desired solution is searching in the form

$$f(z) = \Phi(z) - \frac{1}{\pi}\int_B \left[\frac{\omega(t)}{t-z} + \frac{z\omega(t)}{1-z\overline{t}} - \frac{\omega(t)}{t-1} - \frac{z\omega(t)}{1-\overline{t}}\right] dxdy. \tag{2}$$

Substituting (2) in the Beltrami equation, we get for ω an integral equation for which, by the Fredholm theory, has the unique solution of the Sobolev class $W^{1,p}_{loc}(B)$, for some $p > 2$, see [1], Section 4. If $\varphi(z)$ is continuous only, then the solution to the Dirichlet problem (1) may be obtained as a limit of the uniformly convergent sequence $f_n(z) \to f(z)$ where $f_n(z)$ stands for the unique solution of the Dirichlet problem with sufficiently smooth functions $\varphi_n(z)$ converging to $\varphi(z)$ uniformly on ∂B, see Section 8 in [1]. If the unit disk is replaced by a Jordan domain, then the corresponding Dirichlet problem can be reduced to the unit disk by means of an appropriate conformal mapping. Summing up the above considerations and taking into account Theorem VI.2.2 and the point VI.2.3 in [3], we arrive at the following statement.

Proposition 1. *Let D, 0 ∈D, be a Jordan domain in the complex plane* C *and* $\varphi : \partial D \to \mathrm{R}$ *be a nonconstant continuous function. If* $|\mu(z)|+|\nu(z)| \le k < 1$*, then the Dirichlet problem* (1) *has the unique regular solution normalized by* Imf(0) = 0. *This solution has the representation*

$$f = A\circ g\circ R \qquad (3)$$

where $R : D \to \mathrm{B}$, $R(0) = 0$, *is a conformal mapping and* $g : \overline{\mathrm{B}} \to \mathrm{B}$ *stands for a homeomorphic generalized solution of the quasilinear equation*

$$g_{\bar\varsigma} = \mu^*(\zeta)\cdot g_\varsigma + \upsilon^*(\zeta)\cdot \frac{\overline{A'(g(\varsigma))}}{A'(g(\varsigma))}\cdot g_\varsigma \qquad (4)$$

in B *normalized by* $g(0) = 0$, $g(1) = 1$. *Here* $\mu^* = \frac{R'}{\overline{R'}}\cdot \mu\circ R^{-1}$, $\nu^* = \nu\circ R^{-1}$ *and*

$$A(w) := \frac{1}{2\pi i}\int_{|\omega|=1} \varphi\left(R^{-1}\left(g^{-1}(\omega)\right)\right)\cdot\frac{\omega + wd\omega}{\omega - wd\omega} \qquad (5)$$

is an analytic function in the unit disk B.

The main goal of this paper is to give a brief overview of recent criteria on the existence of regular solutions for the Dirichlet problem (1) in an arbitrary Jordan domain $D \subset$ C when the condition of uniform ellipticity is replaced by the condition that $|\mu(z)| + |\nu(z)| < 1$ almost everywhere in D. The degeneracy of the ellipticity for the Beltrami equations

$$f_{\bar z} = \mu(z)\cdot f_z + \nu(z)\cdot \overline{f_z} \qquad (6)$$

will be controlled by the dilatation coefficient

$$K_{\mu,\nu}(z) := \frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|} \in L^1_{loc}\,. \qquad (7)$$

The solvability of the Dirichlet problem for the degenerate Beltrami
equation

$$f_{\bar z} = \mu(z)\cdot f_z\,, \qquad (8)$$

i.e., when $\nu(z) = 0$, is given in [4–6].

Recall that the problem on existence of homeomorphic solutions for the equation (8) was resolved for the uniformly elliptic case when $\|\mu_\infty\| < 1$ long ago, see e.g. [1,3,7]. The existence problem for the degenerate Beltrami equations (8) when $K_\mu \notin L^\infty$ is currently an active area of research, see e.g. the monographs [5, 8, 9] and the surveys [10, 11], and further references therein. A series of criteria on the existence of regular solutions for the Beltrami equation (6) were given in our recent papers [12–14]. There we called a homeomorphism $f \in W^{1,1}_{loc}(D)$ by a *regular solution* of (6) if f satisfies (6) a.e. in D and $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 \ne 0$ a.e. in D.

2. Existence theorems.

Following [15], we say that a function $u : D \to \mathrm{R}$ has *finite mean oscillation* at a point $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}}\int_{B(z_0,\varepsilon)}\left|u(z) - \tilde u_\varepsilon(z_0)\right| dxdy < \infty, \qquad (9)$$

where

$$\tilde u_\varepsilon(z_0) = \int_{B(z_0,\varepsilon)} u(z)\, dxdy$$

is the mean value of the function $u(z)$ over the disk $B(z_0,\varepsilon)$ with small $\varepsilon > 0$. We also say that a function $u : D \to \mathrm{R}$ is of *finite mean oscillation* in D, abbr. $u \in FMO(D)$ or simply $u \in FMO$, if (1) holds at every point $z_0 \in D$.

Clearly BMO ∈ FMO, where BMO stands for the well-known class of real-valued functions u in a domain $D \subset \mathbb{C}$ with bounded mean oscillation, introduced by John and Nirenberg [16]. There exist examples showing that FMO is not BMO_{loc}, see e.g. [5]. By definition $FMO \subset L^1_{loc}$ but FMO is not a subset of L^p_{loc} for any $p > 1$ in comparison with $BMO_{loc} \subset L^p_{loc}$ for all $p \in [1,\infty)$.

Everywhere further we assume that the functions μ and $\nu : D \to \mathbb{C}$ are extended by zero outside of the domain D.

Theorem 1. *Let D be a Jordan domain in* $\mathbb{C}$ *with* $0 \in D$ *and let* μ *and* $\nu : D \to \mathbb{C}$ *be measurable functions such that* $K_{\mu,\nu}(z) \le Q(z) \le FMO$. *Then the Dirichlet problem* (1) *has a regular solution f with* Imf(0) = 0 *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

As well known, the Dirichlet problem for the partial case when $\mu(z) = 0$,

$$\begin{cases} f_{\bar z} = \nu(z)\cdot \overline{f_z}, & z \in D,\\ \lim\limits_{z\to\zeta} \mathrm{Re}\, f(z) = \varphi(\zeta), & \forall\, \zeta \in \partial D, \end{cases} \qquad (10)$$

takes a key part in applied mathematics (heat/electrical conductivity, magnetic permeability, elastic stiffness, etc), see, i.e., [17–19] and the references therein.

As a consequence of Theorem 1, we obtain the following statement.

Corollary 1. *If*

$$\overline{\lim_{\varepsilon\to 0}}\int_{B(z_0,\varepsilon)}\frac{1+|\nu(z)|}{1-|\nu(z)|}dxdy < \infty \qquad \forall\, z_0 \in D \quad (11)$$

Then the Dirichlet problem (2) *in a Jordan domain D,* $0 \in D$, *has a regularsolution f with* Im f(0) = 0 *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

Theorem 2. *Let D be a Jordan domain in* $\mathbb{C}$ *with* $0 \in D$ *and let* μ *and* $\nu : D \to \mathbb{C}$ *be measurable functions such that* $K_{\mu,\nu} \in L^1_{loc}(D)$. *Suppose that*

$$\int_{\varepsilon<z-z_0<\varepsilon_0} K_{\mu,\nu}(z)\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \quad \forall\, z_0 \in D \quad (12)$$

as $\varepsilon \to 0$ *for some* $\varepsilon_0 = \delta(z_0)$. *Then the Dirichlet problem* (1) *has a regularsolution f with* Im f(0) = 0 *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

The condition (12) can be replaced by the weaker condition

$$\int\limits_{\varepsilon<z-z_0<\varepsilon_0} \frac{K_{\mu,\nu}(z)dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \quad (13)$$

as well as by similar conditions in terms of iterative logarithms.

Theorem 3. *Let D be a Jordan domain in* $\mathbb{C}$ *with* $0\in D$ *and let* $\mu,\nu : D \to \mathbb{B}$ *be measurable functions,* $K_{\mu,\nu} \in L^1(D)$ *and* $k_{z_0}(r)$ *be the mean value of* $K_{\mu,\nu}(z)$ *over the circle* $|z - z_0| = r$. *Suppose that*

$$\int\limits_0^{(z_0)} \frac{dr}{rk_{z_0}(r)} = \infty \qquad \forall\, z_0 \in D. \quad (14)$$

Then the Dirichlet problem (1) *has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

Corollary 2. *In particular, the conclusion of Theorem 3 holds if*

$$k_{z_0}(r) = O\left(\log\frac{1}{r}\right) \quad \text{as} \quad r\to 0 \qquad \forall\, z_0 \in D\ . \quad (15)$$

In fact, the condition (7) can be replaced by the similar conditions in terms of iterative logarithms.

Immediately on the basis of Theorem 3.1 in [20] or Theorem 3.17 in [21], we obtain the next significant consequence of Theorem 3.

Theorem 4. *Let D be a Jordan domain in* $\mathbb{C}$ *with* $0\in D$ *and let* μ *and* $\nu : D \to \mathbb{C}$ *be measurable functions such that*

$$\int\limits_D K_{\mu,\nu}(z)dxdy < \infty \quad (16)$$

where $\Phi : [0,\infty] \to [0,\infty]$ *is a non-decreasing convex function with the condition*

$$\int\limits_\delta^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \quad (17)$$

for some $\delta > \Phi(1)$. *Then the Dirichlet problem* (1) *has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

Corollary 3. *In particular, the conclusion holds if some* $\alpha > 0$ then

$$\int\limits_D e^{\alpha K_{\mu,\nu}(z)}dxdy < \infty\ . \quad (18)$$

By the Stoilow factorization theorem, see e.g. [22], every regular solution *f* to the Dirichlet problem

$$\begin{cases} f_{\bar z} = \mu(z)\cdot f_z, \quad z\in D, \\ \lim\limits_{z\to\varsigma} \operatorname{Re} f(z) = \varphi(\varsigma), \quad \forall\, \varsigma \in \partial D, \end{cases} \quad (19)$$

has the representation $f = h\circ g$ where $g : D \to \mathbb{B}$ stands for a homeomorphic $W^{1,1}_{loc}$ solution to the Beltrami equation $g_{\bar z} = \mu(z)\cdot g_z$, and $h : \mathbb{B} \to \mathbb{C}$ is analytic. By Theorem 5.50 from [21] the condition (17) is not only sufficient but also necessary to have a homeomorphic $W^{1,1}_{loc}$ solution for all such Beltrami equations with the integral constraint

$$\int\limits_D K_\mu(z)dxdy < \infty\ . \quad (20)$$

Note also that in the above theorem we may assume that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex and non–decreasing on the whole segment $[0,\infty]$ but only on a segment $[T,\infty]$ for some $T \in (1,\infty)$.

Let us consider the partial case of the Beltrami equation (14) when

$$\mu(z) = \nu(z) = \frac{\lambda(z)}{2}. \quad (21)$$

The equation of the form

$$f_{\bar z} = \lambda(z)\operatorname{Re} f_z \quad (22)$$

with $|\lambda(z)| < 1$ a.e. is called the *reduced Beltrami equation*, considered e.g. in [23] and [24], though the term was not introduced there. We write

$$K_\lambda(z) := \frac{1+|\lambda(z)|}{1-|\lambda(z)|}\ . \quad (23)$$

Theorem 5. *Let D be a Jordan domain in* $\mathbb{C}$ *with* $0\in D$ *and let* $\lambda : D \to \mathbb{C}$ *be a measurable function such that*

$$\int\limits_D \Phi\left(K_\lambda(z)\right)dxdy < \infty\,, \quad (24)$$

where $\Phi : [0,\infty] \to [0,\infty]$ *is a non-decreasing convex function with the condition* (9)*. Then the Dirichlet problem*

$$\begin{cases} f_{\bar z} = \lambda(z)\cdot \operatorname{Re} f_z, \qquad z\in D, \\ \lim\limits_{z\to\zeta} \operatorname{Re} f(z) = \varphi(\zeta), \quad \forall\, \zeta \in \partial D, \end{cases} \quad (25)$$

in a Jordan domain D, $0\in D$*, has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

Finally, on the basis of Corollary 3, we obtain the following consequence.

Corollary 4. *In particular, the conclusion holds if some* $\alpha > 0$, then

$$\int\limits_D e^{\alpha K_\lambda(z)}dxdy < \infty\ . \quad (26)$$

The above results remain true for the case in (6) when

$$\nu(z) = \mu(z)\, e^{i\vartheta(z)} \quad (27)$$

with an arbitrary measurable function $\theta(z) : D \to \mathbb{R}$ and, in particular, for the equations of the form

$$f_{\bar z} = \lambda(z)\operatorname{Im} f_z \quad (28)$$

with a measurable coefficient $\lambda : D \to \mathbb{C}$, $|\lambda(z)| < 1$ a.e., see e.g. [23].

Complete proofs of the criteria for existence of regular solutions for the Dirichlet problem (1) in a Jordan domain $D\in\mathbb{C}$, given in this section, the reader can find in the original paper [25], see also [26]. For the proofs we make use of the approximate procedure based on the Proposition 1, the convergence theorems for the Beltrami equations (6) when $K_{\mu,\nu} \in L^1_{loc}$ established in [13], and the Stoilow factorization of open discrete mappings.

The Schwarz formula

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi i} \int_{|\zeta|=1} \operatorname{Re} f(\zeta) \cdot \frac{\zeta + z \, d\zeta}{\zeta - z \, d\zeta}, \quad (29)$$

that allows to recover an analytic function *f* in the unit disk B by its real part $\varphi(\zeta) = \operatorname{Re} f(\zeta)$ on the boundary of B up to a purely imaginary additive constant $c = i\operatorname{Im} f(0)$ as well as the Arzela–Askoli theorem combined with moduli techniques are also used.

REFERENCES

1. Bojarski B. Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients. *Mat. Sb*. – 1957. – v.43(85), no. 4. – P. 451–503 [in Russian]; transl. in *Rep. Univ. Jyväskylä, Dept. Math. Stat*. – 2009. – v.118. – P. 1–64.
2. Vekua I.N. *Generalized analytic functions*. Pergamon Press, London. - 1962.
3. Lehto O., Virtanen K. Quasiconformal Mappings in the Plane. Springer, New York etc. - 1973.
4. Dybov Yu. On regular solutions of the Dirichlet problem for the Beltrami equations. *Comp. Var. Ell. Equ*. – 2010. – v. 55(12). – P. 1099–1116.
5. Gutlyanskii V., Ryazanov V., Srebro U., Yakubov E. The Beltrami Equation: A Geometric Approach. *Developments in Mathematics*, *Vol.26*. Springer, New York, 2012.
6. Kovtonyuk D., Petkov I., Ryazanov V. On the Dirichlet problem for the Beltrami equations in finitely connected domains. *Ukr. Mat. Zh*. – 2012. – v. 64. – P. 932–944.
7. Ahlfors L.V. *Lectures on Quasiconformal Mappings*. D. Van Nostrand Company, Inc., Princeton etc. - 1966.
8. Astala K., Iwaniec T. and Martin G.J. Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. *Princeton Math. Ser., Vol.48*. Princeton Univ. Press, Princeton. - 2009.
9. Martio O., Ryazanov V., Srebro U., Yakubov E. *Moduli in Modern Mapping Theory*. Springer Monographs in Mathematics, Springer, New York etc. - 2009.
10. Gutlyanskii V., Ryazanov V., Srebro U., Yakubov E. On recent advances in the degenerate Beltrami equations. *Ukr. Mat. Visn*. – 2010. – v.7, N4. – P. 467–515 [in Ukrainian]; transl. in *J. Math. Sci*. – 2011. – v. 175, N 4. – P. 413–449.
11. Srebro U., Yakubov E. *The Beltrami equation. Handbook in Complex Analysis: Geometric function theory, Vol. 2*. Elseiver B. V. - 2005. – P. 555–597.
12. Bojarski B., Gutlyanskii V., Ryazanov V. General Beltrami equations and BMO. *Ukr. Mat. Visn*. – 2008. – v. 5, N3. – P. 305-326 [in Ukrainian]; transl. in *Math. Bull*. – 2008. – v. 5, N3. – P. 305–326.
13. Bojarski B., Gutlyanskii V., Ryazanov V. On Beltrami equations with two characteristics. *Comp. Var. Ell. Equ*. – 2009. – v. 54. – P. 933–950.
14. Bojarski B., Gutlyanskii V., Ryazanov V. On Integral Conditions for the General Beltrami Equations. *Comp. Anal. Oper. Theory*. – 2011. – v. 5, N3. – P. 835–845.
15. Ignat'ev A., Ryazanov V. Finite mean oscillation in the mapping theory. *Ukr. Mat. Visn*. – 2005. – v. 2, N3. – P. 395–417 [in Russian]; transl. in *Ukr. Math. Bull*. – 2005. – v. 2, N3. – P. 403–424.
16. John F., Nirenberg L. On functions of bounded mean oscillation. *Comm. Pure Appl. Math*. – 1961. – v. 14. – P. 415–426.
17. Krushkal' S.L., Kühnau R. *Quasiconformal mappings: new methods and applications*. Nauka, Novosibirsk. - 1984 [in Russian].
18. Hamilton D.H. Area distortion of quasiconformal mappings. *Handbook in complex analysis: Geometric function theory, Vol. 1*. North-Holland, Amsterdam. - 2002. – P. 147-160.
19. Astala K., Nesi V. Composites and quasiconformal mappings: new optimal bounds in two dimensions. *Calc. Var*. – 2003. – v. 18. – P. 335–355.
20. Ryazanov V., Srebro U., Yakubov E. Integral conditions in the mapping theory. *Ukr. Mat. Visn*. – 2010. – v. 7, N1. – P. 73–87; transl. in *Math. Sci. J*. – 2011. – v. 173, N4. – P. 397–407.
21. Ryazanov V., Srebro U., Yakubov E. Integral conditions in the theory of the Beltrami equations. *Comp. Var. Ell. Equ*. – 2012. – v. 57, N12. – P. 1247–1270.
22. Stoilow S. *Lecons sur les Principes Topologue de le Theorie des Fonctions Analytique*. Gauthier-Villars, 1938. Riemann, Gauthier-Villars, Paris, 1956 [in French].
23. Bojarski B. Primary solutions of general Beltrami equations. *Ann. Acad. Sci. Fenn. Math*. – 2007. – v. 32, N2. – P. 549–557.
24. Volkovyskii L.I. *Quasiconformal mappings*. L'vov Univ. Press. - 1954 [in Russian].
25. Bojarski B., Gutlyanskii V., Ryazanov V. Dirichlet problem for general Beltrami equations in Jordan domains. *Ukr. Mat. Visn*. – 2012. – v. 9, N4. – P. 460-476 [in Ukrainian]; transl. in *J. Math. Sci*. - 2013. – v. 190, N4. – P.525-39.
26. Bojarski B., Gutlyanskii V., Ryazanov V. On Dirichlet problem for Beltrami equations with two characteristics. *www.arxiv.org, arXiv*:1211.0425v [math.CV] 2 Nov 2012, 16 pp.