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ABOUT MAPPINGS IN ORLICZ-SOBOLEV CLASSES ON RIEMANNIAN MANIFOLDS

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A survey of recent results on the continuous and homeomorphic extension to the boundary of homeomorphisms with finite distortion between domains on Riemannian manifolds in the Orlicz-Sobolev classes $W^{1,\varphi}_{loc}$ under a condition of the Calderon type for the function φ and, in particular, in the Sobolev classes $W^{1,p}_{loc}$ under $p > n-1$ is given in the paper.

KEY WORDS: Orlicz-Sobolev classes, homeomorphisms, Riemannian manifolds, Sobolev classes.

ОБ ОТОБРАЖЕНИЯХ КЛАССОВ ОРЛИЧА-СОБОЛЕВА НА РИМАНОВЫХ МНОГООБРАЗИЯХ

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В работе представлен обзор последних результатов о непрерывных гомеоморфных расширениях к границе гомеоморфизмов с конечной дисторсией между областями Римановых многообразий в классах Орлича-Соболева $W^{1,\varphi}_{loc}$ при условиях типа Каледрона для функции φ и, в частности, в классах Соболева $W^{1,p}_{loc}$ при $p > n-1$.

КЛЮЧЕВЫЕ СЛОВА: классы Орлича-Соболева, гомеоморфизмы, многообразия Римана, классы Соболева.

ПРО ВІДІБРАЖЕННЯ КЛАСІВ ОРЛІЧА-СОБОЛЄВА НА РИМАНОВИХ МНОГОВИДАХ

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В роботі наведений огляд останніх результатів о безперервних гомеоморфних розширеннях до границь гомеоморфізмів зі скінченими дисторсіями між областями Ріманових многовидів в класах Орліча-Соболєва $W^{1,\varphi}_{loc}$ за умовами типу Каледрона для функції φ і, зокрема, в класах Соболєва $W^{1,p}_{loc}$ при $p > n-1$.

КЛЮЧОВІ СЛОВА: класи Орлича-Соболєва, гомеоморфізмом, многовиди Рімана, класи Соболєва.

1. Introduction. We give here a survey of our results from the recent paper [1] where we proved many theorems on the continuous and homeomorphic extension to the boundary of the so-called lower Q-homeomorphisms between domains on Riemannian manifolds. On this basis, we also formulated the corresponding consequences for homeomorphisms with finite distortion in the Orlicz-Sobolev classes $W^{1,\phi}_{loc}$ under a condition of the Calderon type for the function ϕ and, in particular, in the Sobolev classes $W^{1,p}_{loc}$ for $p > n-1$. The latter is the main content of the present survey.

Recall some definitions concerning the theory of manifolds (see, e.g., [2]-[5]). An n-*dimensional topological manifold* M^n is a Hausdorff topological space with countable basis in which every point has an open neighborhood that is homeomorphic to R^n. By *a map on the manifold* M^n, we call a pair (U,ϕ), where U is an open subset of the space M^n, and ϕ is the homeomorphic mapping of the subset U onto an open subset of the coordinate space R^n. This mapping puts every point $p \in U$ in the bijective correspondence to a collection of n numbers, which are its *local coordinates*. *A smooth manifold* is a manifold with maps (U_α, ϕ_α) whose local coordinates are connected one to another in a smooth (C^∞) manner.

By the Riemannian manifold (M^n, g), we call a smooth manifold together with a Riemannian metric given on it, i.e., with a positive definite symmetric tensor field $g = g_{ij}(x)$ given in coordinate maps with the transformation rule

$$`g = g_{ij}(x) = g_{kl}(y(x))\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}.$$

In what follows, we assume that the tensor field $g_{ij}(x)$ is also smooth. *A length element* on (M^n,g) is given by the invariant differential form $ds^2 = g_{ij}(x)dx^i dx^j := \sum_{i,j=1}^{n} g_{ij}dx^i dx^j$, where g_{ij} is the metric tensor, and x^i are the local coordinates. The *geodesic distance* $d(p_1,p_2)$ is defined as the infimum of lengths of piecewise smooth curves connecting the points p_1 and p_2 in (M^n,g) (see [3, p. 94]). We also recall that a *volume element* on (M^n,g) is determined by the invariant form $dv = \sqrt{\det g_{ij}}\,dx^1...dx^n$ (see, e.g., [5]). We note that $\det g_{ij} > 0$ by virtue of the positive definiteness of g_{ij} (see, e.g., [6]).

Let D be a domain on the Riemannian manifold (M^n,g), $n \ge 2$. Given a convex increasing function $\phi:[0,\infty)\to[0,\infty)$, $\phi(0)=0$, the symbol L^ϕ denotes the *Orlicz space* of all functions $f: D \to R$ such that $\int_D \phi\left(\frac{|f(x)|}{\lambda}\right) dv(x) < \infty$ for some $\lambda > 0$.

By the *Orlicz-Sobolev class* $W^{1,\phi}_{loc}(D)$, we call the class of all locally integrable functions f given in D with the first generalized derivatives (in local coordinates) whose gradient ∇f belongs locally in the domain D to the Orlicz space L^ϕ. We note that $W^{1,\phi}_{loc} \subset W^{1,\phi}_{loc}$. by definition. As usual, we write $f \in W^{1,p}_{loc}$, if $\phi(t) = t^p$, $p \ge 1$.

If f is a locally integrable vector-function of n real variables $x_1,...,x_n$, $f = (f_1,...,f_m)$, $f_i \in W^{1,1}_{loc}$, $i = 1,...,m$, and, on any compact set $C \subset D$

$$\int_C \phi\left(\frac{|\nabla f(x)|}{\lambda}\right) dv(x) < \infty,$$

for some $\lambda > 0$ where

$$|\nabla f(x)| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)^2},$$

we also write $f \in W^{1,\phi}_{loc}$.

We use also the notation $W^{1,\phi}_{loc}$ in the case of the mappings $f: D \to D_*$ between domains D and D_* on Riemannian manifolds with different dimensions and for functions ϕ which are more general than those in the Orlicz classes, where the convexity of the function ϕ was always *a priori* assumed. Note that the Orlicz-Sobolev classes are intensively studied at present in various aspects (see, e.g., references in [7]-[9]).

2. FMO functions. Let (M^n,g) be a Riemannian manifold, $n \ge 2$. Similarly to [10], cf. also [11]-[13], we say that a function $\phi: M^n \to R$ has *finite mean oscillation at a point* $x_0 \in M^n$, denoted $\phi \in FMO(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{v(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |\phi(x) - \tilde{\phi}_\varepsilon|\, dv(x) < \infty,$$

$$\forall x_0 \in M^n,$$

where $\tilde{\phi}_\varepsilon = \frac{1}{v(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} \phi(x)\, dv(x)$ is the mean value of the function $\phi(x)$ over the geodesic ball $B(x_0,\varepsilon)$ with respect to the measure v.

Proposition 2.1. *If for some collection of numbers,* $\phi_\varepsilon \in R$, $\varepsilon \in (0,\varepsilon_0)$,

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{v(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |\phi(x) - \phi_\varepsilon|\, dv(x) < \infty,$$

then $\phi \in FMO(x_0)$.

Corollary 2.1. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{v(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} |\phi(x)|\, dv(x) < \infty,$$

then $\phi \in FMO(x_0)$.

3. Regular boundaries. First, we recall required definitions from [12-14]. They say that a domain D is *locally connected in a point* $x_0 \in \partial D$ if for any neighborhood U of a point x_0 there exists a neighborhood $V \subset U$ of the point x_0 such that $V \cap D$ is connected. It is known that the Jordan domains are locally connected at each boundary point.

The definitions also say that the boundary of a domain D is *strongly accessible at a point* $x_0 \in \partial D$ if for any neighborhood U of the point x_0 there exist a compact set $E \subset D$, a neighborhood $V \subset U$ of the point x_0, and a number $\delta > 0$ such that

$$M(\Delta(E,F;D)) \ge \delta$$

for any continuum F in D intersecting ∂U and ∂V.

The definitions also say that the boundary ∂D is *weakly flat at a point* $x_0 \in \partial D$ if for any number $P > 0$ and a neighborhood U of the point x_0 there is its neighborhood $V \subset U$ such that

$$M(\Delta(E,F;D)) \ge P$$

for any continuums E and F in D intersecting ∂U and ∂V.

Moreover, the boundary ∂D is called *strongly accessible* and *weakly flat* if it is such at every its point.

Proposition 3.1. *If* ∂D *is weakly flat at a point* $x_0 \in \partial D$*, then* ∂D *is strongly accessible from* D *at the point* x_0.

Lemma 3.1. *If* ∂D *is weakly flat at a point* $x_0 \in \partial D$*, then* D *is locally connected in* x_0.

Remark 3.1. Finally, we note that all known regular domains on Riemannian manifolds as smooth, Lipschitzian, convex, uniform and QED-domains (quasiextremal distance domains by Gehring-Martio, see [15]) have weakly flat and, hence, strongly accessible boundaries, and are locally connected on their boundaries (see [16]). Thus, the results of the present work can be applied to all above-mentioned regular domains.

4. On the boundary behavior of mappings in $W^{1,\phi}_{loc}$**.**

Finally, we present the corresponding results concerning the boundary behavior of homeomorphisms with finite distortion of the Orlicz-Sobolev classes $W^{1,\phi}_{loc}$ between domains D and D_* on smooth Riemannian manifolds (M^n, g) and (M^n_*, g^*), $n \ge 3$.

Here $J(x,f) := \lim\limits_{r\to 0} \dfrac{v_*(f(B(x,r)))}{v(B(x,r))}$ a

$$L(x,f) := \limsup_{y\to x} \frac{d_*(f(x),f(y))}{d(x,y)}.$$

Moreover, we set $K(x,f) = L^n(x,f)/J(x,f)$ if $J(x,f) \ne 0$, $K(x,f) = 1$ if $L(x,f) = 0$, and $K(x,f) = \infty$ at the rest of points.

Theorem 4.1. *Let* D *be locally connected on the boundary, let* $\overline{D}$ *be compact, and let* ∂D_* *be weakly flat. If* $f : D \to D_*$ *is a homeomorphism with finite distortion of the Orlicz-Sobolev class* $W^{1,\phi}_{loc}$ *under condition*

$$\int_1^\infty \left[\frac{t}{\phi(t)}\right]^{\frac{1}{n-2}} dt < \infty, \tag{1}$$

and $K(x,f) \in L^{n-1}(D)$*, then* f^{-1} *has a continuous extension to* $\overline{D_*}$.

In what follows, we assume that the function $K(x,f)$ is extended by zero outside of the domain D.

Theorem 4.2. Let D *be locally connected at a point* $x_0 \in \partial D$*, let* ∂D_* *be strongly accessible and let* $\overline{D_*}$ *be a compact. Then any homeomorphism with finite distortion* $f : D \to D_*$ *of the Orlicz-Sobolev class* $W^{1,\phi}_{loc}$ *under condition (1) and* $K^{n-1}(x,f) \in FMO(x_0)$ *is extended at the point* x_0 *by continuity on* (M^n_*, g^*).

Corollary 4.1. *Let* D *be locally connected at a point* $x_0 \in \partial D$*, let* ∂D_* *be strongly accessible and let* $\overline{D_*}$ *be a compact. Then any homeomorphism with finite distortion* $f : D \to D_*$ *of the Orlicz-Sobolev class* $W^{1,\phi}_{loc}$ *under condition (1) and*

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{v(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} K^{n-1}(x)dv(x) < \infty$$

is extended to the point x_0 *by continuity on* (M^n_*, g^*).

Theorem 4.3. *Let* D *be locally connected on its boundary, let* ∂D_* *be weakly flat, and let* $\overline{D}$ *and* $\overline{D_*}$ *be compact. Then any homeomorphism* $f : D \to D_*$ *of the Orlicz-Sobolev class* $W^{1,\phi}_{loc}$ *under condition (1) and* $K^{n-1}(x,f) \in FMO$ *admits a homeomorphic extension* $\overline{f} : \overline{D} \to \overline{D_*}$.

Theorem 4.4. *Let* D *be locally connected at a point* $x_0 \in \partial D$*, let* ∂D_* *be strongly accessible, let* $\overline{D_*}$ *be compact, and let* $f : D \to D_*$ *be a homeomorphism with finite distortion of the Orlicz-Sobolev class* $W^{1,\phi}_{loc}$ *under condition (1). If*

$$\int_0^{\delta(x_0)} \frac{dr}{\|K_f\|_{n-1}(x_0,r)} = \infty, \tag{2}$$

where $0 < \delta(x_0) < d(x_0) = \sup\limits_{x\in D} d(x,x_0)$ *is such that* $B(x_0, \delta(x_0))$ *is a normal neighborhood of the point* x_0 *and* $\|K_f\|_{n-1}(x_0,r) = \left(\int_{S(x_0,r)} K^{n-1}(x,f)dA\right)^{\frac{1}{n-1}}$,

the f *has the extension to the point* x_0 *by continuity on* (M^n_*, g^*)*. If, additionally, (2) holds for all points* $x_0 \in \partial D$*,* D *is locally connected on its boundary,* $\overline{D}$ *is compact,* ∂D_* *is weakly flat, and* $K(x,f) \in L^{n-1}(D)$*, then* f *has a homeomorphic extension* $\overline{f} : \overline{D} \to \overline{D_*}$.

Corollary 4.2. *Let* D *be locally connected on the boundary, let* ∂D_* *be strongly accessible, let* $\overline{D_*}$ *be compact, and let* $f : D \to D_*$ *be a homeomorphism with finite distortion of the Orlicz-Sobolev class* $W^{1,\phi}_{loc}$ *under condition (1) and*

$$\int_D \Phi\,(K^{n-1}(x,f))\,dv(x) < \infty \tag{3}$$

for a convex increasing function $\Phi : [0,\infty] \to [0,\infty]$ *such that*

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau\left(\Phi^{-1}(\tau)\right)^{\frac{1}{n-1}}} = \infty \qquad (4)$$

for some $\delta > \Phi(0)$. *Then* f *is extended to the point* x_0 *by continuity. If, additionally,* D *is locally connected everywhere on its boundary,* $\overline{D}$ *is compact, and* ∂D_* *is weakly flat, then* f *admits a homeomorphic extension* $\overline{f} : \overline{D} \to \overline{D_*}$.

Remark 4.1. All these results hold, in particular, for homeomorphisms with finite distortion of the Sobolev class $W^{1,p}_{loc}$ for $p > n-1$, as well as for homeomorphisms of the class $W^{1,1}_{loc}$ with $K_f \in L^q_{loc}$ for $q > n-1$. We note also that condition (4) is not only sufficient but also necessary for the continuous extension on the boundary of homeomorphisms of the Sobolev class $W^{1,1}_{loc}$ with $K_f \in L^q_{loc}$ for $q > n-1$ and with the integral conditions (3) on $K(x,f)$ (see the example in Lemma 5.1 [17]). See also Remark 3.1 above.

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